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Design of fault-tolerant control for MTTF

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SUMMARY

Mean time to failure (MTTF) is an important reliability index of fault-tolerant control systems, which is chosen as a design objective in this paper. However, it is usually evaluated from stochastic reliability models, and no *analytical* expression is available to relate MTTF to controller parameters. To overcome this difficulty, a two-stage design scheme is proposed in this paper: A gradient-based search is firstly carried out on probabilistic H_∞ performance characteristics for MTTF requirement; a sequential randomized algorithm with a weighted violation function is then developed for a controller design to satisfy the required H_∞ performance, and its convergence is guaranteed with probability 1. Two iterative algorithms are carried out alternately to implement this scheme, and a controller can be designed for MTTF requirement.

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1. INTRODUCTION

In order to meet high reliability requirement of safety-critical processes, extensive progress has been made in fault-tolerant control systems (FTCSs) [1]. Reliability, the main motivation for developing FTCSs, is a major concern in their analysis and design. For example, Wu studied a Markov process model built from serial–parallel block diagrams [2]; Walker proposed a semi-Markov model by defining its states as the mode combinations of faults and fault detection and isolation (FDI) schemes [3]. However, these models provide rough descriptions only but some

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unique characteristics of FTCSs are ignored. In a previous work [4], we have developed a new reliability index and its evaluation method, incorporating the dynamical characteristics of FTCSs. In this paper, we focus on the design of FTCSs using this reliability criterion.

The reliability-based design of FTCSs is basically an optimization problem of designing a controller to achieve desired reliability index. A reliability-based reconfiguration strategy was recently developed for FTCSs in [5] by optimizing system structure to improve reliability, but the effects of control actions were not considered. In the active control of civil engineering structures, reliability-based design is usually converted to covariance control or classical optimization problems using approximate reliability measures [6, 7]. Similarly, reliable control aims at guaranteeing stability and/or control performance under component faults [8]. In FTCSs, a valid reliability index is usually evaluated from stochastic models and cannot be readily converted to a control objective. Owing to the numerical procedures of building and solving stochastic reliability models, it is generally difficult to write the reliability index as an analytical function of controller parameters. In order to overcome this difficulty, stabilizing controller parameterization and randomization-based methods were developed in [9] to find the statistically optimal controller with respect to reliability. However, the parameterization method is restricted to specific types of models, hence, the result cannot be extended to general forms.

This paper discusses a new controller design method to optimize a long-run reliability index, mean time to failure (MTTF). This index is evaluated based on probabilistic control performance characteristics, which are used to relate controller to MTTF. The basic idea is to perform MTTF optimization in two stages: (1) a gradient-based search is performed on control performance characteristics that are updated along the fastest increasing direction of MTTF; (2) the updated control performance characteristics are then transmitted to a controller design algorithm, which updates controller accordingly to satisfy these control performance characteristics. Each design stage is completed by one iterative algorithm, and two algorithms are carried out alternately to complete controller design. This two-stage design overcomes the difficulty caused by the nonexistence of analytical objective functions of MTTF with respect to controller parameters.

The control performance is characterized by a probabilistic H_∞ criterion, defined as the probability that the H_∞ norm is within the specified threshold when assuming bounded random uncertainties. H_∞ norm is suitable for describing long-term static control performance; when transient behaviors are of interest, a model-matching structure can be adopted to represent transient performance using H_∞ norm. To design a controller for this probabilistic H_∞ criterion, a sequential randomized algorithm is adopted. This algorithm iteratively updates controller based on uncertainty samples and is effective to handle probabilistic robust performance. For example, it has been used for robust guaranteed cost control [10], robust linear matrix inequities problem [11], linear parameter varying design [12], and searching for common Lyapunov functions [13]. In this paper, probabilistic H_∞ control is considered, and the main difference from previous work lies in the introduction of a weighted composite violation function to handle multiple regime models in FTCSs; both state feedback and two-degree-of-freedom (2DOF) controls are discussed; and both the convexity of violation function and the convergence of algorithms are proved for this new problem.

The remainder of this paper is organized as follows: System model is introduced in Section 2; performance characterization and reliability index are presented in Section 3; controller design algorithms are discussed in Sections 4 and 5 for state feedback control and 2DOF control, respectively; Section 6 addresses output feedback controller design when state information is unavailable; and an example is finally given in Section 7 to demonstrate the method.

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2. SYSTEM MODEL

Consider the following Markovian dynamical model of FTCSs with modeling uncertainties [14, p. 32]:

$$
\begin{aligned}
\dot{x}(t) &= A(\zeta(t),\Delta)x(t)+B(\zeta(t),\Delta)u(\eta(t),t)+E(\zeta(t),\Delta)w(t) \\
z(t) &= C(\zeta(t),\Delta)x(t)+D(\zeta(t),\Delta)u(\eta(t),t)+F(\zeta(t),\Delta)w(t)
\end{aligned}
\tag{1}
$$

where $x(t)\in\mathbb{R}^n, z(t)\in\mathbb{R}^m, u(\eta(t),t)\in\mathbb{R}^p$, and $w(t)\in\mathbb{R}^q$ denote system state, regulated output representing control performance, control input, and exogenous input, respectively. $\mathbb{R}^n$ denotes the real vector space with dimension n. A, B, C, D, E, and F denote system matrices with compatible dimensions determined by discrete modes $\zeta(t)$ and $\eta(t)$, and affected by uncertainty parameter Δ. $\zeta(t)$ and $\eta(t)$ are assumed to be two continuous-time Markov processes. $\Delta\in\mathbb{R}^l$ is assumed to be a random vector with known probability distribution in a bounded set Ω, and the entries in system matrices are affected by the elements in Δ.

Remark 1

Different from a measured output, $z(t)$ is the regulated output to characterize control performance. For example, in tracking control, $z(t)$ can be taken as the tracking error between controlled output and reference command input. $w(t)$ contains exogenous inputs to the system, such as reference command input and disturbances, whose effects on $z(t)$ are undesirable and to be suppressed by designing controllers. It may represent unknown disturbance or noise, but can also represent known reference input. In the aircraft example in Section 7, $w(t)$ represents known command input from pilot. Hence, $w(t)$ cannot be assumed as a random signal in general. As a result, model (1) cannot be written as an Ito stochastic differential equation with a Gaussian white-noise input.

$\zeta(t)$ represents system fault modes and is assumed to be a homogeneous Markov process with a finite state space $S_1=\{0,1,\ldots,N_1\}$, $N_1\in\mathbb{N}$, where $\mathbb{N}$ denotes the set of nonnegative integers. In FTCSs, $\zeta(t)$ is usually unknown, so its estimate $\eta(t)$ provided by an FDI scheme determines control input $u(\eta(t),t)$. $\eta(t)$ is assumed to be a conditionally Markov process with state space $S_2=\{0,1,\ldots,N_2\}$ to describe FDI results, $N_2\in\mathbb{N}$.

Remark 2

The required assumption to describe FDI mode $\eta(t)$ as a Markov process is the memoryless Markov property [15, p. 233]. According to [16, Section 2.1], if FDI schemes are designed based on single sample hypothesis tests and the noise statistics are white, this assumption is valid. For general FDI schemes, it is difficult to check this memoryless assumption based on their designs, and semi-Markov processes were used instead in [17]. If FDI history data are available for estimating empirical sojourn time distribution, the assumption can be tested by checking whether sojourn time follows exponential distribution or not. Under the assumption that FDI modes can be modeled by a semi-Markov process, the exponential distribution implies that a Markov process is a valid model [15, p. 316]; considering that semi-Markov assumption is usually acceptable for describing FDI modes, this sojourn time distribution test can also be used in general to check Markov assumption.

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The transition probability of $\zeta(t)$ from mode i to j, $i, j \in S_1$, in the infinitesimal time interval of δt, is given by

$$\zeta(t): p_{ij}(\delta t) = \begin{cases} \alpha_{ij}\delta t + o(\delta t), & i \neq j \\ 1 + \alpha_{ii}\delta t + o(\delta t), & i = j \end{cases}$$

where $\alpha_{ij} \geqslant 0$ and $\alpha_{ii} = -\sum_{j \in S_1, j \neq i} \alpha_{ij}$ denote the transition rates of $\zeta(t)$, and $o(\delta t)$ denotes higher-order infinitesimal terms. When $\zeta(t) = k, k \in S_1$, the transition probability of $\eta(t)$ from mode i to j, $i, j \in S_2$, in the infinitesimal time interval of δt is given by

$$\eta(t): p^k_{ij}(\delta t) = \begin{cases} \beta^k_{ij}\delta t + o(\delta t), & i \neq j \\ 1 + \beta^k_{ii}\delta t + o(\delta t), & i = j \end{cases}$$

where $\beta^k_{ij} \geqslant 0$ and $\beta^k_{ii} = -\sum_{j \in S_2, j \neq i} \beta^k_{ij}$ represent the transition rates of $\eta(t)$ given $\zeta(t) = k$. In general, these transition rates, α_{ij} and β^k_{ij}, compose the generator matrices of $\zeta(t)$ and $\eta(t)$, respectively, denoted by $H_\zeta = [\alpha_{ij}]_{N_1 \times N_1}$ and $H^k_\eta = [\beta^k_{ij}]_{N_2 \times N_2}$.

Remark 3

Model (1) is a linear dynamical system subject to Markovian switchings and has been discussed in many references. According to [14, p. 32, 18, p. 117, 19, p. 143], the existence conditions of a unique solution are the Lipschitz and linear growth conditions of the right-hand side of (1) with respect to x. For a general setting, a rigorous proof is provided in [20, p. 81] for stochastic differential equations with Markovian switchings, and model (1) can be deemed as one of its special cases.

A static state feedback controller in a switching structure is considered for FTCSs. Here, static means that the controller is a pure gain. If the exogenous input $w(t)$ contains unknown disturbances only, the controller is composed of a set of state feedback gains, denoted by $\mathbf{K} \triangleq \{K_j, j \in S_2\}$, and $u(\eta(t), t) = K_j x(t)$ when $\eta(t) = j$. With this controller, the closed-loop system equations become

$$\begin{aligned} \dot{x}(t) &= [A(\zeta(t), \Delta) + B(\zeta(t), \Delta) K_{\eta(t)}] x(t) + E(\zeta(t), \Delta) w(t) \\ z(t) &= [C(\zeta(t), \Delta) + D(\zeta(t), \Delta) K_{\eta(t)}] x(t) + F(\zeta(t), \Delta) w(t) \end{aligned} \tag{2}$$

where $K_{\eta(t)}$ represents K_j when $\eta(t) = j$.

On the other hand, if the exogenous input contains known reference command input, the controller may be in a 2DOF structure, denoted by $\mathscr{K} \triangleq \{(K_j, L_j), j \in S_2\}$, and $u(\eta(t), t) = K_j x(t) + L_j w(t)$ when $\eta(t) = j$. The term 2DOF means that there are two control gains involved for state feedback and reference feedforward, respectively. The closed-loop system equations become

$$\begin{aligned} \dot{x}(t) &= [A(\zeta(t), \Delta) + B(\zeta(t), \Delta) K_{\eta(t)}] x(t) + [E(\zeta(t), \Delta) + B(\zeta(t), \Delta) L_{\eta(t)}] w(t) \\ z(t) &= [C(\zeta(t), \Delta) + D(\zeta(t), \Delta) K_{\eta(t)}] x(t) + [F(\zeta(t), \Delta) + D(\zeta(t), \Delta) L_{\eta(t)}] w(t) \end{aligned} \tag{3}$$

The closed-loop system (2) or (3) contains two discrete modes $\zeta(t)$ and $\eta(t)$, also referred to as system regime modes. For fixed regime modes $\zeta(t) = i$ and $\eta(t) = j$, (2) or (3) is reduced to a linear uncertain system, and the transfer function from $w(t)$ to $z(t)$ is denoted by $G_{ij}(s, \Delta)$, called a regime model. Hence, (2) or (3) represents a collection of linear uncertain regime models

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denoted by $\{G_{ij}(s,\Delta), i\in S_1, j\in S_2\}$. Owing to possible incorrect FDI decisions, each controller K_j or (K_j, L_j) may be used for N_1+1 possible regime models: $G_{0j}(s,\Delta),\ldots,G_{N_1 j}(s,\Delta)$, $j\in S_2$. This is the major difference from jump linear systems, where the number of controllers equals that of regime models [21].

3. RELIABILITY INDICES

3.1. Control performance characterization

The control performance of $G_{ij}(s,\Delta)$ is assumed to be represented by a model-based criterion, such as system norms. Let $\varpi(G_{ij}(s,\Delta))$ denote the performance measure calculated for fixed regime modes $\zeta(t)=i$, $\eta(t)=j$, and a particular uncertainty sample Δ. The allowable performance bound when $\zeta(t)=i$ is denoted by ρ_i. To take into account the random uncertainty Δ, a probabilistic performance description is considered for each regime model:

$$\gamma_{ij}\triangleq \Pr\{\varpi(G_{ij}(s,\Delta))\leqslant\rho_i\},\quad i\in S_1,\quad j\in S_2 \tag{4}$$

For fixed $\zeta(t)=i$ and $\eta(t)=j$, probabilistic performance γ_{ij} can be estimated by

$$\gamma_{ij}\approx\frac{1}{N}\sum_{h=1}^{N} 1_{\varpi(G_{ij}(s,\Delta_h))\leqslant\rho_i} \tag{5}$$

where Δ_h denotes the generated uncertainty sample according to its distribution, and $G_{ij}(s,\Delta_h)$ the corresponding closed-loop transfer function. The indicator function $1_{\varpi(G_{ij}(s,\Delta_h))\leqslant\rho_i}$ equals 1 if $\varpi(G_{ij}(s,\Delta_h))\leqslant\rho_i$ and 0 otherwise. N can be determined based on the allowable estimation error using statistical theory, such as Chernoff's bound [10]. If N is large enough, the estimation error can be ignored, and (5) can be deemed as the true probabilistic performance γ_{ij}.

Model-based criteria are mainly defined for steady-state or long-term performance. When regime modes are under fast transitions, transient performance is of interest and should also be reflected in control performance characterization. In this paper, H_∞ norm is selected as the model-based criterion and can represent transient performance by using suitable weighting functions [22]. However, adjusting weighting functions may need trial and error, and an alternative method is adopted here based on model-matching H_∞ design [23]. Its basic idea is shown in Figure 1: the required transient performance is represented by a desired model, and the controller is designed to minimize the H_∞ norm from reference input to mismatch error signal. The reference input can be chosen as the exogenous input $w(t)$ and mismatch error as $z(t)$. In this way, H_∞ controller can be designed for transient performance. The controller may take the 2DOF state feedback structure, and the closed-loop equations have similar forms as (3).

Remark 4

Model matching control structure effectively captures transient dynamic characteristics of one regime model when the states of $\zeta(t)$ and $\eta(t)$ are fixed. During the transitions of $\zeta(t)$ and $\eta(t)$, plant dynamics are switched from one regime model to another, which may also cause undesired transients. However, they are not directly considered in this controller design. In the reliability model of FTCSs, a hard deadline concept was borrowed to account for theses transients [4]. Its main idea is to allow performance to be unsatisfactory for a maximum time during mode transitions.

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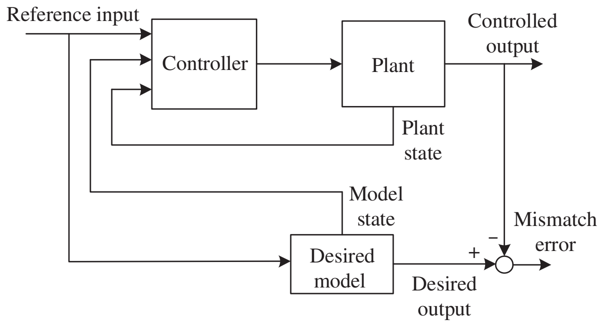

Figure 1. Model-matching diagram.

This hard deadline may be deemed as an implicit or indirect consideration of these transients in the reliability index.

3.2. Reliability definition

Definition 1 (*Li et al.* [*4*])
The reliability function $R(t)$ is the probability that, during time interval $[0,t]$, FTCSs either satisfy control objectives or violate them only temporally for a short time no longer than the presumed hard deadline T_{hd}.

$R(t)$ is a function criterion, and an alternative scalar index, MTTF, is often preferable in controller design. It is defined as the expected lifetime of satisfactory operation:

$$\mathrm{MTTF}=\int_0^\infty R(t)\,\mathrm{d}t$$

These reliability criteria provide quantitative measures on overall long-term performance of FTCSs. To avoid high costs of emergency repairs between periodic maintenance activities, the probability of failure within a maintenance period should be reduced to a certain level. For this purpose, the interested problem is to design a controller that achieves suboptimal MTTF exceeding $\overline{\mathrm{MTTF}}$, where $\overline{\mathrm{MTTF}}$ represents minimum MTTF requirement and can be determined based on maintenance period.

For the sake of reliability evaluation, a semi-Markov process $X_{\mathrm{R}}(t)$ was constructed in [4]. Its state space S_{R} is composed of operational or up states and a unique down state. The transition characteristics of $X_{\mathrm{R}}(t)$ is defined by its semi-Markov kernel $Q(X_k,X_h,t)$ based on probabilistic performance γ_{ij}, where X_k and X_h represent the states of $X_{\mathrm{R}}(t)$. The detailed definition and derivation of $Q(X_k,X_h,t)$ can be found in [4]. Based on $X_{\mathrm{R}}(t)$, MTTF can be calculated by [24]

$$\mathrm{MTTF}=p_0^{\mathrm{T}}(I-P_{\mathrm{up}})^{-1}\mu \qquad (6)$$

where I denotes the identity matrix with compatible dimensions, p_0 the vector of initial probability distribution of $X_{\mathrm{R}}(t)$, P_{up} its limiting transition probability matrix, and μ the vector of expected

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sojourn time at up states of $X_R(t)$. The elements of these three parameters are defined by

$$p_0(X_k) = \Pr\{X_R(0) = X_k\}$$
$$P_{up}(X_k, X_h) = \lim_{t\to\infty} Q(X_k, X_h, t)$$
$$\mu(X_k) = \int_0^\infty \left(1 - \sum_{X_l \in S_R} Q(X_k, X_l, t)\right) t\,dt$$

where $X_k, X_h \in S_R$, and both are up states. If $I - P_{up}$ is not invertible, MTTF$=\infty$, which is generally not achievable in practice. In the sequel, $I - P_{up}$ is assumed to be invertible.

Owing to the construction of $X_R(t)$, it is difficult to establish the analytical relation between controller and MTTF. Considering that MTTF is calculated from the parameters of X_R which is constructed based on control performance characteristics γ_{ij}, γ_{ij} can be used as a connection parameter between controller and MTTF. Based on (6), the derivative of MTTF with respect to γ_{ij} can be calculated by

$$\frac{\mathrm{dMTTF}}{\mathrm{d}\gamma_{ij}} = p_0^{\mathrm{T}}(I - P_{up})^{-1}\frac{\mathrm{d}P_{up}}{\mathrm{d}\gamma_{ij}}(I - P_{up})^{-1}\mu + p_0^{\mathrm{T}}(I - P_{up})^{-1}\frac{\mathrm{d}\mu}{\mathrm{d}\gamma_{ij}} \tag{7}$$

In the set of controllers $\mathbf{K} = \{K_j, j \in S_2\}$ or $\mathscr{K} = \{(K_j, L_j), j \in S_2\}$, each K_j or (K_j, L_j) is designed for $N_1 + 1$ regime models and therefore determines $N_1 + 1$ probabilistic performance parameters γ_{ij}, $i = 0, \ldots, N_1$. For K_j or (K_j, L_j), define the gradient of MTTF as

$$\nabla\mathrm{MTTF}_j \triangleq \frac{\left[\dfrac{\mathrm{dMTTF}}{\mathrm{d}\gamma_{0j}} \cdots \dfrac{\mathrm{dMTTF}}{\mathrm{d}\gamma_{N_1 j}}\right]^{\mathrm{T}}}{\sqrt{\sum_{i\in S_1}\left(\dfrac{\mathrm{dMTTF}}{\mathrm{d}\gamma_{ij}}\right)^2}} \tag{8}$$

which is composed of the derivatives of MTTF with respect to probabilistic parameters related to K_j or (K_j, L_j), $i \in S_1$, $j \in S_2$. With $\nabla\mathrm{MTTF}_j$ available, the following gradient-based iterative search algorithm is adopted for MTTF optimization, where $\mathbf{K}^l \triangleq \{K_j^l, j \in S_2\}$ and $\mathscr{K}^l \triangleq \{(K_j^l, L_j^l), j \in S_2\}$ represent the state feedback and 2DOF controllers, respectively, at the lth iteration.

Algorithm 1 (*MTTF optimization*)

1. Initialization: Set $l = 0$; select minimum reliability requirement $\overline{\mathrm{MTTF}}$ and step size $\tau > 0$; randomly generate the initial value of controller $\mathbf{K}^0$ or $\mathscr{K}^0$; and estimate probabilistic performance γ_{ij}^0 using (5).
2. At iteration l, calculate MTTF based on controller $\mathbf{K}^l$ or $\mathscr{K}^l$. If MTTF$>\overline{\mathrm{MTTF}}$, stop and the controller at current iteration satisfies MTTF requirement; otherwise, if $\sqrt{\sum_{i\in S_1}(\frac{\mathrm{dMTTF}}{\mathrm{d}\gamma_{ij}})^2} < \varepsilon$, a small positive number, stop because the algorithm is at a local optimum but $\overline{\mathrm{MTTF}}$ is not achieved.

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3. For each $j\in S_2$, calculate ∇MTTF_j^l using (7)–(8); use Algorithm 2 to obtain $\mathbf{K}^{l+1}$ or $\mathscr{K}^{l+1}$ such that $\gamma_{ij}^{l+1}\geqslant\gamma_{ij}^{l}+\tau\nabla\text{MTTF}_{ij}^l$ for any $i\in S_1$ and $j\in S_2$, where ∇MTTF_{ij}^l denotes the element of ∇MTTF_j^l.
4. Go to step 2 and start the new iteration $l+1$.

Remark 5
In Algorithm 1, γ_{ij} is iterated along the gradient direction of MTTF, and its value is used to direct controller update. Because the convexity of MTTF with respect to γ_{ij} is not guaranteed, the gradient search may run into a local optimum, and a controller for the required MTTF cannot be found. This problem always exists when using gradient search for a nonconvex problem. It can be handled by the change of initial values or the relaxation of $\overline{\text{MTTF}}$. In step 3, the controller is designed to satisfy the probabilistic performance using Algorithm 2 presented in the following section.

4. SEQUENTIAL RANDOMIZED ALGORITHMS FOR STATE FEEDBACK CONTROL

This section considers the design of a state feedback controller. For notational simplicity, for $\zeta(t)=i,\eta(t)=j,i\in S_1,j\in S_2$, denote $A_i(\Delta)\triangleq A(\zeta(t),\Delta)$, $B_i(\Delta)\triangleq B(\zeta(t),\Delta)$, $C_i(\Delta)\triangleq C(\zeta(t),\Delta)$, $D_i(\Delta)\triangleq D(\zeta(t),\Delta)$, $E_i(\Delta)\triangleq E(\zeta(t),\Delta)$, $F_i(\Delta)\triangleq F(\zeta(t),\Delta)$, $\overline{A}_{ij}(\Delta)\triangleq A(\zeta(t),\Delta)+B(\zeta(t),\Delta)K_j$, and $\overline{C}_{ij}(\Delta)\triangleq C(\zeta(t),\Delta)+D(\zeta(t),\Delta)K_j$. For fixed $\zeta(t)=i$ and $\eta(t)=j$, (2) is reduced to a linear uncertain system

$$G_{ij}:\begin{cases}\dot{x}(t)=\overline{A}_{ij}(\Delta)x(t)+E_i(\Delta)w(t)\\ z(t)=\overline{C}_{ij}(\Delta)x(t)+F_i(\Delta)w(t)\end{cases}\tag{9}$$

Let $G_{ij}(s,\Delta)$ denote the transfer function from $w(t)$ to $z(t)$. Its H_∞ norm $\|G_{ij}(s,\Delta)\|_\infty$ is selected as the performance criterion, and the probabilistic performance is reduced to $\gamma_{ij}=\Pr\{\|G_{ij}(s,\Delta)\|_\infty\leqslant\rho_i\}$. The following lemma then provides a sufficient condition to check whether $\|G_{ij}(s,\Delta)\|_\infty\leqslant\rho_i$. In the sequel, Δ is not shown in system matrices for notational simplicity.

Lemma 1
For system (9), assume that the initial state $x(0)=0$ and $\rho_i^2 I-F_i^{\mathrm{T}}F_i>0$, where I denotes an identity matrix with compatible dimensions. For fixed i, j, and a particular uncertainty sample Δ, $\|G_{ij}(s)\|_\infty\leqslant\rho_i$ holds if there exists $P_{ij}\geqslant 0$ such that

$$(\overline{A}_{ij})^{\mathrm{T}}P_{ij}+P_{ij}\overline{A}_{ij}+(\overline{C}_{ij})^{\mathrm{T}}\overline{C}_{ij}+(P_{ij}E_i+(\overline{C}_{ij})^{\mathrm{T}}F_i)(\rho_i^2I-F_i^{\mathrm{T}}F_i)^{-1}(E_i^{\mathrm{T}}P_{ij}+F_i^{\mathrm{T}}\overline{C}_{ij})\leqslant 0\tag{10}$$

The proof is standard by using a quadratic Lyapunov function as given in [25, p. 212]. Owing to Lemma 1, if inequality (10) holds with probability γ_{ij} when Δ varies probabilistically, K_j satisfies probabilistic performance $\gamma_{ij}=\Pr\{\|G_{ij}(s,\Delta)\|_\infty\leqslant\rho_i\}$. K_j can be designed using a sequential randomized algorithm presented in this section.

The following notations are adopted in this section: The space of real symmetric matrices is a Hilbert space with the inner product $\langle M,N\rangle\triangleq\text{Tr}(M^{\mathrm{T}}N)$ and Frobenius norm $\|M\|\triangleq(\sum_{i,j=1}^{n}M(i,j))^{1/2}$, where $\text{Tr}(\cdot)$ denotes the trace of a matrix and n the dimension of M. For a real symmetric matrix M, its projection onto the convex cone of nonnegative definite matrices is defined as

$$M^+ \triangleq \arg\min_{N\geqslant 0} \|M-N\|$$

M^+ can be computed explicitly as follows [10]: If $M=U\Lambda U^{\mathrm{T}}$, where U is orthogonal and Λ is diagonal with entries $\lambda_1,\ldots,\lambda_n$, then $M^+=U\Lambda^+U^{\mathrm{T}}$, where Λ^+ is diagonal with entries $\max\{0,\lambda_1\},\ldots,\max\{0,\lambda_n\}$.

4.1. Violation function and gradient computation

In this subsection, the matrix inequality (10) in Lemma 1 is converted to a scalar convex function. Let us begin with a special case that $D_i=F_i=0$, and the left-hand side of (10) is simplified and denoted as V_{ij}, $i\in S_1$, $j\in S_2$:

$$V_{ij}\triangleq A_i^{\mathrm{T}}P_{ij}+P_{ij}A_i+K_j^{\mathrm{T}}B_i^{\mathrm{T}}P_{ij}+P_{ij}B_iK_j+P_{ij}E_iE_i^{\mathrm{T}}P_{ij}/\rho_i^2+C_i^{\mathrm{T}}C_i\leqslant 0 \tag{11}$$

Let f denote a functional on the space of symmetric matrices that assigns matrix M a real number $f(M)$. The gradient of $f(M)$ is denoted as $\partial_{\mathrm{R}}f(M)$, meaning

$$f(M+\delta M)=f(M)+\langle\partial_M f(M),\delta M\rangle+o(\|\delta M\|)$$

where δM denotes a small perturbation in M. $f(M)$ is convex if and only if [26, p. 69, Chapter 3]

$$f(M+\delta M)\geqslant f(M)+\langle\partial_M f(M),\delta M\rangle$$

Lemma 2 (*Liberzon and Tempo* [*13*])
The functional $f(M)\triangleq\frac{1}{2}\|M^+\|^2$ is convex and differentiable with gradient given by $\partial_M f(M)=M^+$.

Using Lemma 2, a violation function of (11) is defined as

$$\begin{aligned}v_{ij}(K_j,P_{ij},\Delta)\triangleq f(V_{ij})=\tfrac{1}{2}\|(&A_i^{\mathrm{T}}P_{ij}+P_{ij}A_i+K_j^{\mathrm{T}}B_i^{\mathrm{T}}P_{ij}+P_{ij}B_iK_j\\&+P_{ij}E_iE_i^{\mathrm{T}}P_{ij}/\rho_i^2+C_i^{\mathrm{T}}C_i)^+\|^2\end{aligned} \tag{12}$$

where $i\in S_1$ and $j\in S_2$. Obviously, $v_{ij}(K_j,P_{ij},\Delta)\geqslant 0$, and $v_{ij}(K_j,P_{ij},\Delta)=0$ if and only if $V_{ij}\leqslant 0$. In other words, (11) holds if and only if $v_{ij}(K_j,P_{ij},\Delta)=0$.

Lemma 3
$v_{ij}(K_j,P_{ij},\Delta)$ is convex in K_j and P_{ij}, respectively, and its gradients with respect to these two matrix variables are

$$\partial_{K_j}v_{ij}(K_j,P_{ij},\Delta)=2B_i^{\mathrm{T}}P_{ij}V_{ij}^+$$

$$\begin{aligned}\partial_{P_{ij}}v_{ij}(K_j,P_{ij},\Delta)=&(B_iK_j+A_i+E_iE_i^{\mathrm{T}}P_{ij}/\rho_i^2)V_{ij}^+\\&+V_{ij}^+(K_j^{\mathrm{T}}B_i^{\mathrm{T}}+A_i^{\mathrm{T}}+P_{ij}E_iE_i^{\mathrm{T}}/\rho_i^2)\end{aligned}$$

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The proofs are given in Appendix A. For the general cases that $D_i \neq 0$ and $F_i \neq 0$, the gradients are given as follows, which can be proved in a similar way:

$$\partial_{K_j} v_{ij}(K_j, P_{ij}, \Delta)$$

$$= 2[B_i^{\mathrm{T}} P_{ij} + D_i^{\mathrm{T}} C_i + D_i^{\mathrm{T}} F_i (\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} E_i^{\mathrm{T}} P_{ij}^{\mathrm{T}}$$

$$+ (D_i^{\mathrm{T}} D_i + D_i^{\mathrm{T}} F_i (\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} F_i^{\mathrm{T}} D_i) K_j] V_{ij}^{+}$$

$$\partial_{P_{ij}} v_{ij}(K_j, P_{ij}, \Delta)$$

$$= [A_i + B_i K_j + E_i (\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} F_i^{\mathrm{T}} (C_i + D_i K_j) + E_i (\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} E_i^{\mathrm{T}}] V_{ij}^{+}$$

$$+ V_{ij}^{+} [A_i^{\mathrm{T}} + K_j^{\mathrm{T}} B_i + (C_i + D_i K_j)^{\mathrm{T}} F_i (\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} E_i + E_i (\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} E_i^{\mathrm{T}}]$$

Owing to false alarms, the FDI estimate $\eta(t)$ may be different from $\zeta(t)$. As a result, given fixed $j \in S_2$, each controller gain K_j may appear in N_1+1 inequalities, $V_{ij} \leqslant 0$ for $i = 0, 1, \ldots, N_1$. To take these N_1+1 inequalities into account simultaneously, a weighted composite violation function is defined as

$$\psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = \sum_{i=0}^{N_1} \theta_{ij} v_{ij}(K_j, P_{ij}, \Delta) \tag{13}$$

where θ_{ij} denotes a positive weight corresponding to inequality $V_{ij} \geqslant 0$ for $\zeta(t) = i, i \in S_1, j \in S_2$.

Lemma 4
Given $j \in S_2$, if $\theta_{ij} > 0$ for all $i \in S_1$, $\psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}) = 0$ is equivalent to $V_{ij} \leqslant 0$ simultaneously for all $i \in S_1$.

Proof
As $\theta_{ij} > 0$ and $v_{ij}(K_j, P_{ij}, \Delta) \geqslant 0$, $\psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = 0$ if and only if $v_{ij} = 0$ holds simultaneously for $i = 0, 1, \ldots, N_1$, $j \in S_2$. Based on the definition of v_{ij} in (12), $v_{ij}(K_j, P_{ij}, \Delta) = 0$ if and only if $V_{ij} \leqslant 0$, which concludes the proof. □

Lemma 4 shows that ψ_j can be used as a composite violation function for multiple matrix inequalities if all weights are positive. Hence, θ_{ij} can be selected freely among all positive values.

Lemma 5
Given $j \in S_2$, $\psi_j(K_j, P_{0j}, \ldots, P_{N_1 j})$ is convex in K_j and P_{ij}, $i \in S_1$, and its gradients are given by

$$\partial_{K_j} \psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = \sum_{i=0}^{N_1} \theta_{ij} \partial_{K_j} v_{ij}(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) \tag{14}$$

$$\partial_{P_{ij}} \psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = \theta_{ij} \partial_{P_{ij}} v_{ij}(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) \tag{15}$$

Lemma 5 is obvious considering (13) and the properties of gradient and convexity.

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4.2. *Controller design algorithm*

Let S^j_{KP} represents the robust solution set of (13) defined as

$$S^j_{KP} \triangleq \{(K_j, P_{0j}, P_{1j}, \ldots, P_{N_1 j}) : \psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = 0, \forall \Delta \in \Omega\} \tag{16}$$

Two standard assumptions of sequential algorithms are made here as follows [10]:

Assumption 1
The solution set S^j_{KP} defined in (16) contains a nonempty interior for any given $j \in S_2$.

Assumption 2
If $(K_j, P_{0j}, P_{1j}, \ldots, P_{N_1 j}) \notin S_{KP}$, $\Pr\{\psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) > 0\} > 0$. Based on Assumption 1, there exist an interior point $(K^\#_j, P^\#_{0j}, \ldots, P^\#_{N_1 j}) \in S^j_{KP}$ and a ball $B_{r_j} \subset S^j_{KP}$ centered at $(K^\#_j, P^\#_{0j}, \ldots, P^\#_{N_1 j})$. The knowledge of radius r_j of B_{r_j} can be used to determine step size in the algorithm presented in this section.

In Algorithm 1, the control design algorithm is to find $(K^{l+1}_j, P^{l+1}_{0j}, \ldots, P^{l+1}_{N_1 j})$ such that $\gamma^{l+1}_{ij} \geqslant \gamma^l_{ij} + \tau \nabla \mathrm{MTTF}^l_{ij}$, where $i \in S_1$, $j \in S_2$, and $l \in \mathbb{N}$ represent the iteration index of Algorithm 1. The algorithm is in an iterative structure: At iteration $k \in \mathbb{N}$, if the violation function $\psi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k) > 0$ for a randomly generated uncertainty sample Δ^k, K^{k+1}_j and P^{k+1}_{ij} are updated by

$$K^{k+1}_j = K^k_j - \mu^k_j \frac{\partial_{K_j} \psi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)}{\phi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)} \tag{17}$$

$$P^{k+1}_{ij} = \left[P^k_{ij} - \mu^k_j \frac{\partial_{P_{ij}} \psi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)}{\phi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)} \right]^+ \tag{18}$$

where ϕ_j represents the overall size of the gradient in Lemma 5:

$$\phi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k) \triangleq (\|\partial_{K_j} \psi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)\|^2 + \sum_{i=0}^{N_1} \|\partial_{P_{ij}} \psi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)\|^2)^{1/2}, \quad j \in S_2 \tag{19}$$

μ^k_j denotes the step size calculated by

$$\mu^k_j \triangleq \frac{\psi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)}{\phi_j(K^k_j, P^k_{0j}, \ldots, P^k_{N_1 j}, \Delta^k)} + r_j \tag{20}$$

where $r_j > 0$ denotes the radius of $B_{r_j} \subset S_{KP}$ centered at $(K^\#_j, P^\#_{0j}, \ldots, P^\#_{N_1 j})$.

Remark 6
In this paper, r_j is assumed to be a known priori for choosing step size μ^k_j in the sequential algorithm. If r_j is not known, classical choice of step size in stochastic gradient algorithms can

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be used for μ_j^k. For example, $\lim_{k\to\infty}\mu_j^k=0$ and $\sum_{k=0}^{\infty}\mu_j^k=\infty$ [12, 27]. More discussions on this issue can be found in [28].

Note that the projection operation is used in (18) to ensure that P_{ij} converge to a nonnegative definite matrix. If the violation function $\psi_j(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)=0$, let $K_j^{k+1}=K_j^k$ and $P_{ij}^{k+1}=P_{ij}^k$, $i\in S_1$, $j\in S_2$.

The controller design algorithm is given in Algorithm 2, where $\gamma_{ij}^{l*}\triangleq\gamma_{ij}^l+\tau\nabla\text{MTTF}_{ij}^l$, i denotes fault mode, j FDI mode, l iteration index in Algorithm 1, and k iteration index in Algorithm 2.

Algorithm 2 (*Controller design for probabilistic performance*)

1. *Initialization*: Set $k=0$, $K_j^{l0}=K_j^l$, and $P_{ij}^{l0}=P_{ij}^l$, taken from iteration l in Algorithm 1, $i\in S_1$, $j\in S_2$.
2. At iteration k, estimate the probabilistic performance γ_{ij}^{lk} of K_j^{lk} using (5) for all $i\in S_1$. If $\gamma_{ij}^{lk}\geqslant\gamma_{ij}^{l*}$ for all $i\in S_1$, stop and return K_j^{lk} to Algorithm 1 as K_j^{l+1}.
3. Determine positive weight θ_{ij}^{lk} based on γ_{ij}^{lk}, γ_{ij}^{l*}, and ∇MTTF_{ij}^l, $i\in S_1$.
4. Generate an uncertainty sample Δ^{lk}; if $\psi_j(K_j^{lk}, P_{0j}^{lk}, \ldots, P_{N_1 j}^{lk}, \Delta^{lk})>0$, update K_j^{lk} and P_{ij}^{lk} using (17) and (18), respectively; then, goto step 2.

As the probabilistic performance requirement γ_{ij}^{l*} is calculated based on the gradient ∇MTTF_{ij}^l, it is ideal to have γ_{ij}^{lk} increase along this gradient direction for fast convergence. Based on Lemma 4, $\psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta)$ is a valid composite violation function as long as the weight $\theta_{ij}^{lk}>0$. Considering that θ_{ij}^{lk} also appear in gradient calculation (14)–(15), the increasing direction of γ_{ij}^{lk} can be adjusted by determining θ_{ij}^{lk} based on heuristic rules, which helps to reduce iteration number [29].

4.3. Convergence result

Theorem 1

If Assumptions 1 holds, iterations (17)–(18) ensure the following inequality:

$$\|K_j^{k+1}-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^{k+1}-P_{ij}^{\#}\|^2\leqslant\|K_j^k-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^k-P_{ij}^{\#}\|^2-r_j^2 \tag{21}$$

where $(K_j^{\#}, P_{0j}^{\#}, \ldots, P_{N_1 j}^{\#})\in S_{KP}$ denotes a robust solution.

Proof

The proof follows a standard procedure in subgradient algorithms [10, 12, 13]. Owing to Assumption 1, define the following feasible solution in S_{KP}:

$$\overline{K}_j=K_j^{\#}+r_j\frac{\partial_{K_j}\psi_j(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)}{\phi_j(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)} \tag{22}$$

$$\overline{P}_{ij}=P_{ij}^{\#}+r_j\frac{\partial_{P_{ij}}\psi_j(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)}{\phi_j(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)}, \quad i\in S_1 \tag{23}$$

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Hence, $\psi_j(\overline{K}_j, \overline{P}_{0j}, \ldots, \overline{P}_{N_1 j}, \Delta)=0$ for all $\Delta \in \Omega$. For notational simplicity, the variables of ψ_j are omitted. If $\psi_j>0$, we have

$$
\begin{aligned}
&\|K_j^{k+1}-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^{k+1}-P_{ij}^{\#}\|^2 \\
&=\left\|K_j^k-K_j^{\#}-\mu_j^k\frac{\partial_{K_j}\psi_j}{\phi_j}\right\|^2+\sum_{i=0}^{N_1}\left\|\left[P_{ij}^k-\mu_j^k\frac{\partial_{K_j}\psi_j}{\phi_j}\right]^+-P_{ij}^{\#}\right\|^2 \\
&\leqslant\left\|K_j^k-K_j^{\#}-\mu_j^k\frac{\partial_{K_j}\psi_j}{\phi_j}\right\|^2+\sum_{i=0}^{N_1}\left\|P_{ij}^k-\mu_j^k\frac{\partial_{K_j}\psi_j}{\phi_j}-P_{ij}^{\#}\right\|^2 \\
&=\|K_j^k-K_j^{\#}\|^2-2\mu_j^k\left\langle\frac{\partial_{K_j}\psi_j}{\phi_j}, K_j^k-\overline{K}_j\right\rangle-2\mu_j^k\left\langle\frac{\partial_{K_j}\psi_j}{\phi_j}, \overline{K}_j-K_j^{\#}\right\rangle+\left\|\mu_j^k\frac{\partial_{K_j}\psi_j}{\phi_j}\right\|^2 \\
&\quad+\sum_{i=0}^{N_1}\left(\|P_{ij}^k-P_{ij}^{\#}\|^2-2\mu_j^k\left\langle\frac{\partial_{P_{ij}}\psi_j}{\phi_j}, P_{ij}^k-\overline{P}_{ij}\right\rangle-2\mu_j^k\left\langle\frac{\partial_{P_{ij}}\psi_j}{\phi_j}, \overline{P}_{ij}-P_{ij}^{\#}\right\rangle+\left\|\mu_j^k\frac{\partial_{P_{ij}}\psi_j}{\phi_j}\right\|^2\right)
\end{aligned}
$$

where the inequality is because of the property of projection operation [10]. On the basis of (19), we have

$$
\left\|\mu_j^k\frac{\partial_{K_j}\psi_j}{\phi_j}\right\|^2+\sum_{i=0}^{N_1}\left\|\mu_j^k\frac{\partial_{P_{ij}}\psi_j}{\phi_j}\right\|^2=(\mu_j^k)^2
$$

Owing to the convexity of ψ_j in K_j and P_{ij}, we have

$$
\left\langle\frac{\partial_{K_j}\psi_j}{\phi_j}, K_j^k-\overline{K}_j\right\rangle\geqslant\frac{\psi_j}{\phi_j}, \quad \left\langle\frac{\partial_{P_{ij}}\psi_j}{\phi_j}, P_{ij}^k-\overline{P}_{ij}\right\rangle\geqslant\frac{\psi_j}{\phi_j}
$$

Because of (19) and (22)–(23), we have

$$
\left\langle\frac{\partial_{K_j}\psi_j}{\phi_j}, \overline{K}_j-K_j^{\#}\right\rangle+\sum_{i=0}^{N_1}\left\langle\frac{\partial_{P_{ij}}\psi_j}{\phi_j}, \overline{P}_{ij}-P_{ij}^{\#}\right\rangle=r_j
$$

Therefore, we have

$$
\|K_j^{k+1}-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^{k+1}-P_{ij}^{\#}\|^2\leqslant\|K_j^k-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^k-P_{ij}^{\#}\|^2+(\mu_j^k)^2-2\mu_j^k\left(\frac{\psi_j}{\phi_j}+r_j\right)
$$

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By substituting μ_j^k defined in (20), we have

$$\|K_j^{k+1}-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^{k+1}-P_{ij}^{\#}\|^2 \leqslant \|K_j^k-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^k-P_{ij}^{\#}\|^2-\left(\frac{\psi_j}{\phi_j}+r_j\right)^2$$

$$\leqslant \|K_j^k-K_j^{\#}\|^2+\sum_{i=0}^{N_1}\|P_{ij}^k-P_{ij}^{\#}\|^2-r_j^2$$

Hence, (21) holds, meaning that the distance to the robust solution is decreasing monotonically. □

Remark 7
Iterations (17)–(18) are originated from subgradient methods, and their convergence is usually proved based on the distance between the decision variables and the solution set [27, p. 25]. Theorem 1 also follows this idea: after each iteration, the distance of controller to robust solution set is reduced by at least r_j^2. Hence, only finite updates are needed before reaching the solution set. Considering there is a positive probability of performing the update based on Assumption 2, this theorem leads to the following convergence result of Algorithm 2.

Proposition 1
If Assumptions 1–2 hold, Algorithm 2 converges in a finite number of iterations with probability 1 to a controller satisfying required probabilistic performance.

Proof
Considering Assumption 2, there is a positive probability of generating an uncertainty sample with $\psi_j(K_j, P_{0j}, \ldots, P_{N_1 j}, \Delta)>0$ and performing iterations (17)–(18) when $(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k)\notin S_{KP}$. In other words, the distance of $(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k)$ to S_{KP} decreases by at least r_j^2 with a positive probability when $(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k)\notin S_{KP}$. Therefore, $(K_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k)$ converges to a robust solution in S_{KP} in a finite number of iterations with probability one, which implies the convergence to a controller with any probabilistic performance. □

5. SEQUENTIAL RANDOMIZED ALGORITHMS FOR 2DOF CONTROL

The design of 2DOF control parallels that of state feedback control. For fixed regime modes $\zeta(t)=i$ and $\eta(t)=j, i\in S_1, j\in S_2$, the closed-loop system (3) is reduced to a linear uncertain system

$$G_{ij}:\begin{cases}\dot{x}(t)=\overline{A}_{ij}x(t)+(E_i+B_iL_j)w(t)\\ z(t)=\overline{C}_{ij}x(t)+(F_i+D_iL_j)w(t)\end{cases} \tag{24}$$

where the simplified notations in Section 4 have been used. Following Lemma 1, its H_∞ norm is not greater than ρ_i if there exists $P_{ij}>0$ such that

$$\overline{A}_{ij}^{\mathrm{T}}P_{ij}+P_{ij}\overline{A}_{ij}+\overline{C}_{ij}^{\mathrm{T}}\overline{C}_{ij}+[P_{ij}(E_i+B_iL_j)+\overline{C}_{ij}^{\mathrm{T}}(F_i+D_iL_j)]$$
$$\times[\rho_i^2I-(F_i+D_iL_j)^{\mathrm{T}}(F_i+D_iL_j)]^{-1}[(E_i+B_iL_j)^{\mathrm{T}}P_{ij}+(F_i+D_iL_j)^{\mathrm{T}}\overline{C}_{ij}]\leqslant 0 \tag{25}$$

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As the controller gain L_j is involved with matrix inverse in this inequality, its convexity is violated when $D_i \neq 0$. Hence, $D_i = 0$ is assumed in order to apply sequential randomized algorithms. Let us begin with the case that $F_i = 0$, and the matrix inequality is reduced to

$$W_{ij} \triangleq A_i^{\mathrm{T}} P_{ij} + P_{ij} A_i + K_j^{\mathrm{T}} B_i^{\mathrm{T}} P_{ij} + P_{ij} B_i K_j + P_{ij} (E_i + B_i L_j)(E_i + B_i L_j)^{\mathrm{T}} P_{ij} / \rho_i^2 + C_i^{\mathrm{T}} C_i \leqslant 0 \quad (26)$$

A violation function of (26) can be defined as

$$w_{ij}(K_j, L_j, P_{ij}, \Delta) \triangleq f(W_{ij}) = \tfrac{1}{2} \| (A_i^{\mathrm{T}} P_{ij} + P_{ij} A_i + K_j^{\mathrm{T}} B_i^{\mathrm{T}} P_{ij} + P_{ij} B_i K_j + P_{ij}(E_i + B_i L_j)(E_i + B_i L_j)^{\mathrm{T}} P_{ij} / \rho_i^2 + C_i^{\mathrm{T}} C_i)^{+} \|^2 \quad (27)$$

Lemma 6

$w_{ij}(K_j, L_j, P_{ij}, \Delta)$ defined in (27) is convex in K_j, L_j, and P_{ij}, respectively, and its gradients with respect to these matrix variables are

$$\partial_{K_j} w_{ij}(K_j, L_j, P_{ij}, \Delta) = 2 B_i^{\mathrm{T}} P_{ij} W_{ij}^{+}$$

$$\partial_{L_j} w_{ij}(K_j, L_j, P_{ij}, \Delta) = 2 B_i^{\mathrm{T}} P_{ij} W_{ij}^{+} P_{ij} (B_i L_j + E_i)$$

$$\partial_{P_{ij}} w_{ij}(K_j, L_j, P_{ij}, \Delta) = [B_i K_j + A_i + (E_i + B_i L_j)(E_i + B_i L_j)^{\mathrm{T}} P_{ij} / \rho_i^2] W_{ij}^{+} + W_{ij}^{+} [K_j^{\mathrm{T}} B_i^{\mathrm{T}} + A_i^{\mathrm{T}} + P_{ij} (E_i + B_i L_j)(E_i + B_i L_j)^{\mathrm{T}} / \rho_i^2]$$

Lemma 6 can be proved in the same way as Lemma 3. This simplified case of $F_i = 0$ corresponds to 2DOF control in model-matching design for transient performance and is of major interest in this paper, as demonstrated in Section 7. The gradients and convexity in the general case of $F_i \neq 0$ can be derived in a similar way and are omitted here for brevity.

Compared with state feedback control in Section 4, 2DOF control involves two control gains (K_j, L_j) and therefore one more decision variable. But the sequential randomized algorithms can be constructed following the same procedures, which are listed as follows without proof. For convenience and comparison purpose, the same notations are used here as in Section 4.

The composite violation function is defined as

$$\psi_j(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = \sum_{i=0}^{N_1} \theta_{ij} w_{ij}(K_j, L_j, P_{ij}, \Delta) \quad (28)$$

and its gradients are

$$\partial_{K_j} \psi_j(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = \sum_{i=0}^{N_1} \theta_{ij} \partial_{K_j} w_{ij}(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) \quad (29)$$

$$\partial_{L_j} \psi_j(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = \sum_{i=0}^{N_1} \theta_{ij} \partial_{L_j} w_{ij}(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) \quad (30)$$

$$\partial_{P_{ij}} \psi_j(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = \theta_{ij} \partial_{P_{ij}} w_{ij}(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta), \quad i = 0, \ldots, N_1 \quad (31)$$

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At kth iteration of randomized algorithm, if the violation function is greater than zero, denoted as $\psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)>0$, K_j^{k+1}, L_j^{k+1}, and P_{ij}^{k+1} are updated by

$$K_j^{k+1} = K_j^k - \mu_j^k \frac{\partial_{K_j} \psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)}{\phi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)} \tag{32}$$

$$L_j^{k+1} = L_j^k - \mu_j^k \frac{\partial_{L_j} \psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)}{\phi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)} \tag{33}$$

$$P_{ij}^{k+1} = \left[P_{ij}^k - \mu_j^k \frac{\partial_{P_{ij}} \psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)}{\phi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)} \right]^+ \tag{34}$$

where

$$\mu_j^k \triangleq \frac{\psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)}{\phi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)} + r_j$$

and

$$\begin{aligned}\phi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k) \triangleq (&\|\partial_{K_j} \psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)\|^2 \\ &+ \|\partial_{L_j} \psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)\|^2 \\ &+ \sum_{i=0}^{N_1} \|\partial_{P_{ij}} \psi_j(K_j^k, L_j^k, P_{0j}^k, \ldots, P_{N_1 j}^k, \Delta^k)\|^2)^{1/2}\end{aligned} \tag{35}$$

$r_j>0$ denotes the radius of S_{KLP} centered at $(K_j^{\#}, L_j^{\#}, P_{0j}^{\#}, \ldots, P_{N_1 j}^{\#})$, and S_{KLP} represents the robust solution set defined as

$$S_{KLP} \triangleq \{(K_j, L_j, P_{0j}, P_{1j}, \ldots, P_{N_1 j}) : \psi_j(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta) = 0, \forall \Delta \in \Omega\} \tag{36}$$

If S_{KLP} contains a nonempty interior for any given $j \in S_2$, iterations (32)–(34) ensure the following inequality:

$$\begin{aligned}&\|K_j^{k+1} - K_j^{\#}\|^2 + \|L_j^{k+1} - L_j^{\#}\|^2 + \sum_{i=0}^{N_1} \|P_{ij}^{k+1} - P_{ij}^{\#}\|^2 \\ &\leqslant \|K_j^k - K_j^{\#}\|^2 + \|L_j^k - L_j^{\#}\|^2 + \sum_{i=0}^{N_1} \|P_{ij}^k - P_{ij}^{\#}\|^2 - r_j^2\end{aligned} \tag{37}$$

Inequality (37) can be derived in a similar way as in Theorem 1, which leads to the convergence of the iterative updates. The 2DOF controller design algorithm can be implemented by replacing iterations (17)–(18) with (32)–(34) in Algorithm 2. And it can be used with Algorithm 1 to find a 2DOF controller for MTTF optimization. If $(K_j, L_j, P_{0j}, P_{1j}, \ldots, P_{N_1 j}) \notin S_{KLP}$ implies $\Pr\{\psi_j(K_j, L_j, P_{0j}, \ldots, P_{N_1 j}, \Delta)>0\}>0$, Algorithm 2 with iterations (32)–(34) is guaranteed to

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converge to a 2DOF controller satisfying required performance with probability 1, which can be proved similarly as in Proposition 1.

Remark 8

Both the state feedback and 2DOF controller require the full information of states and can be designed using sequential randomized algorithms, in which the iterative updates and convergence proof are similar. Their differences lie in the following aspects: (1) The 2DOF design is originated from model-matching problem and needs the knowledge of $w(t)$; (2) it requires the condition $D_j = 0$ to apply the algorithm; (3) it involves one additional feedforward gain L_j, which appears as a new decision variable in the design algorithm.

6. OUTPUT FEEDBACK CONTROL

We consider a general scenario when plant state is not available for controller and the plant output equation is

$$y(t) = U(\zeta(t),\Delta)x(t) + M(\zeta(t),\Delta)u(t) + N(\zeta(t),\Delta)w(t) \tag{38}$$

where $y(t) \in \mathbb{R}^l$ represents measured output, and $U(\zeta(t),\Delta)$, $M(\zeta(t),\Delta)$, and $N(\zeta(t),\Delta)$ system matrices. By using $y(t)$ instead of $x(t)$ in the original 2DOF control, the control input becomes

$$\begin{aligned} u(t) &= K_{\eta(t)} y(t) + L_{\eta(t)} w(t) \\ &= K_{\eta(t)} U(\zeta(t),\Delta)x(t) + K_{\eta(t)} M(\zeta(t),\Delta)u(t) + [K_{\eta(t)} N(\zeta(t),\Delta) + L_{\eta(t)}]w(t) \end{aligned} \tag{39}$$

Obviously, closed-loop system is well-posed if and only if $I - K_{\eta(t)} M(\zeta(t),\Delta)$ is nonsingular, which leads to

$$u(t) = [I - K_{\eta(t)} M(\zeta(t),\Delta)]^{-1} [K_{\eta(t)} U(\zeta(t),\Delta)x(t) + (K_{\eta(t)} N(\zeta(t),\Delta) + L_{\eta(t)})w(t)]$$

Substituting $u(t)$ into (1), the closed-loop system equation becomes

$$\begin{aligned} \dot{x}(t) &= [A(\zeta(t),\Delta) + B(\zeta(t),\Delta)(I - K_{\eta(t)} M(\zeta(t),\Delta))^{-1} K_{\eta(t)} U(\zeta(t),\Delta)]x(t) \\ &\quad + [E(\zeta(t),\Delta) + B(\zeta(t),\Delta)(I - K_{\eta(t)} M(\zeta(t),\Delta))^{-1} (K_{\eta(t)} N(\zeta(t),\Delta) + L_{\eta(t)})]w(t) \\ z(t) &= [C(\zeta(t),\Delta) + D(\zeta(t),\Delta)(I - K_{\eta(t)} M(\zeta(t),\Delta))^{-1} K_{\eta(t)} U(\zeta(t),\Delta)]x(t) \\ &\quad + [F(\zeta(t),\Delta) + D(\zeta(t),\Delta)[I - K_{\eta(t)} M(\zeta(t),\Delta)]^{-1} (K_{\eta(t)} N(\zeta(t),\Delta) + L_{\eta(t)})]w(t) \end{aligned} \tag{40}$$

For fixed regime modes $\zeta(t) = i$ and $\eta(t) = j$, $i \in S_1$, $j \in S_2$, the closed-loop system is reduced to a linear uncertain system

$$G_{ij} : \begin{cases} \dot{x}(t) = [A_i + B_i(I - K_j M_i)^{-1} K_j U_i]x(t) + [E_i + B_i(I - K_j M_i)^{-1}(K_j N_i + L_j)]w(t) \\ z(t) = [C_i + D_i(I - K_j M_i)^{-1} K_j U_i]x(t) + [F_i + D_i(I - K_j M_i)^{-1}(K_j N_i + L_j)]w(t) \end{cases} \tag{41}$$

where the simplified notations in Section 4 have been used. Following Lemma 1, a matrix inequality can be derived in order to have $\|G_{ij}(s,\Delta)\|_\infty \leqslant \rho_i$. However, matrix inverse terms involving controller gains may appear in the inequality and violate convexity. Therefore, $D_i = 0$ and $M_i = 0$

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are assumed in order to apply sequential algorithms, and the matrix inequality for H_∞ performance in Lemma 1 is reduced to

$$W_{ij} \triangleq (A_i + B_i K_j U_i)^{\mathrm{T}} P_{ij} + P_{ij}(A_i + B_i K_j U_i) + C_i^{\mathrm{T}} C_i$$
$$+ [P_{ij}(E_i + B_i K_j N_i + B_i L_i) + C_i^{\mathrm{T}} F_i](\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1}[(E_i + B_i K_j N_i + B_i L_i)^{\mathrm{T}} P_{ij} + F_i^{\mathrm{T}} C_i]$$

Its violation function can be defined as

$$w_{ij}(K_j, L_j, P_{ij}, \Delta) \triangleq \tfrac{1}{2} \| W_{ij} \|^2 \tag{42}$$

Note same notations of violation functions and gradients are used as in Section 5 for comparison purpose.

Lemma 7
$w_{ij}(K_j, L_j, P_{ij}, \Delta)$ defined in (42) is convex in K_j, L_j, and P_{ij}, and its gradients are

$$\partial_{K_j} W_{ij} = 2B_i^{\mathrm{T}} P_{ij} W_{ij}^{+} (P_{ij} E_i + P_{ij} B_i L_j + C_i^{\mathrm{T}} F_i + P_{ij} B_i K_j N_i)(\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} N_i^{\mathrm{T}}$$

$$\partial_{L_j} W_{ij} = 2B_i^{\mathrm{T}} P_{ij} W_{ij}^{+} (P_{ij} B_i L_j + P_{ij} E_i + P_{ij} B_i K_j N_i + C_i^{\mathrm{T}} F_i)(\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1}$$

$$\begin{aligned}\partial_{P_{ij}} W_{ij} = {} & [A_i + B_i K_j U_i + (E_i + B_i K_j N_i + B_i L_i)(\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1} F_i^{\mathrm{T}} C_i \\ & + (E_i + B_i K_j N_i + B_i L_i)(\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1}(E_i + B_i K_j N_i + B_i L_i)^{\mathrm{T}} P_{ij}] W_{ij}^{+} \\ & + W_{ij}^{+} [A_i^{\mathrm{T}} + U_i^{\mathrm{T}} K_j^{\mathrm{T}} B_i^{\mathrm{T}} + C_i^{\mathrm{T}} F_i (\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1}(E_i + B_i K_j N_i + B_i L_i)^{\mathrm{T}} \\ & + P_{ij}(E_i + B_i K_j N_i + B_i L_i)(\rho_i^2 I - F_i^{\mathrm{T}} F_i)^{-1}(E_i + B_i K_j N_i + B_i L_i)^{\mathrm{T}}]\end{aligned} \tag{43}$$

Lemma 7 can be proved in the same way as Lemmas 3 and 6. Once a convex violation function and its gradients are found, the remaining procedures are similar as (28)–(35) and omitted here for brevity. Compared with 2DOF control using state feedback, output 2DOF control contains more complicated calculations of gradients. The corresponding output feedback control of state feedback control in Section 4 is to use only $y(t)$ for controller design, which can be deemed as a special case by making $L_{\eta(t)} = 0$ in (39).

7. EXAMPLE

We consider a demonstration example used in [10] which studies the lateral motion of an aircraft. The plant model under fault-free mode is given by

$$\dot{x}_{\mathrm{p}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & L_{\mathrm{p}} & L_\beta & L_r \\ g/V & 0 & Y_\beta & -1 \\ N_{\dot\beta} & N_{\mathrm{p}} & N_\beta + N_{\dot\beta} Y_\beta & N_r - N_{\dot\beta} \end{bmatrix} x_{\mathrm{p}}(t) + \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} u(t)$$

where the components in state $x_p(t)$ represent, respectively, bank angle, directive of bank angle, sideslip angle, and yaw rate. Two control inputs are rudder and aileron deflections, respectively. The considered faulty mode is the 50% loss of effectiveness in both actuators, represented by the reduction of control input matrices.

The control objective considered here is to make the side slip angle, the third state of $x_p(t)$, track pilot's command, represented by exogenous input $w(t)$. The desired response model from $w(t)$ to side slip angle is represented by a first-order transfer function $2/(s+1)$. This is a typical model-matching problem as illustrated in Figure 1. Let $x_m(t)$ denote the state of the desired model, and $x(t) \triangleq [x_p^T(t)\ x_m(t)]^T$, the augmented state vector. The model-matching problem can be converted to the following standard setup:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & L_p & L_\beta & L_r & 0 \\ g/V & 0 & Y_\beta & -1 & 0 \\ N_{\dot{\beta}} & N_p & N_\beta + N_{\dot{\beta}} Y_\beta & N_r - N_{\dot{\beta}} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w(t) + B(\zeta(t))u(t, \eta(t))$$

$$z(t) = [0\ \ 0\ \ -1\ \ 0\ \ 2]x(t)$$

where $u(t,\eta(t)) = K_{\eta(t)}x(t) + L_{\eta(t)}w(t)$ represents a 2DOF controller in a switching structure. $B(\zeta(t))$ represents the fault effects on control input matrices. Let B_0 and B_1 denote $B(\zeta(t))$ when $\zeta(t)$ is in mode 0 and 1, respectively:

$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1.955 \\ 0.0175 & 0 \\ -1.265 & 0.155 \\ 0 & 0 \end{bmatrix}$$

The modeling uncertainties are introduced by aircraft parameters, and the random vector $\Delta = [L_p\ L_\beta\ L_r\ g/V\ Y_\beta\ N_{\dot{\beta}}\ N_p\ N_\beta\ N_r]^T$. The mean values of these parameters are $L_p = -2.93$, $L_\beta = -4.75$, $L_r = 0.78$, $g/V = 0.086$, $Y_\beta = -0.11$, $N_{\dot{\beta}} = 0.1$, $N_p = -0.042$, $N_\beta = 2.601$, and $N_r = -0.29$. Each parameter is assumed to be perturbed by a relative uncertainty of 10%. For example, L_β is bounded in the interval $[-3.223, -2.637]$. The probability distribution of each parameter is assumed to be a uniform distribution within the corresponding interval.

The fault occurrences and FDI mode transitions are characterized by the generator matrices of $\zeta(t)$ and $\eta(t)$:

$$H_\zeta = \begin{bmatrix} -0.005 & 0.005 \\ 0 & 0 \end{bmatrix}, \quad H_\eta^0 = \begin{bmatrix} -0.2 & 0.2 \\ 2 & -2 \end{bmatrix}, \quad H_\eta^1 = \begin{bmatrix} -2 & 2 \\ 0.2 & -0.2 \end{bmatrix}$$

These parameters can be interpreted as follows: According to H_ζ, the mean occurrence time of faults is 200 s, and the faulty state is absorbing as the components on the second row are all zeros; according to H_η^0, under fault-free mode, the mean time to false alarms is 5 s and mean returning

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time from false alarms is 0.5 s; according to H_η^1, the mean time to missing detections is 5 s and mean returning time is 0.5 s. Hence, this FDI may give frequent incorrect detections, and these parameters are used for illustrative purpose only.

These interpretations have direct relations with classical parameters on diagnostic imperfectness, such as mean time between false alarms and mean time between missing detections. For example, when plant and FDI are both at fault-free modes, mean time to false alarms is the estimate on occurrence time of false alarms starting from correct detections. The corresponding classical concept, mean time between missing detections, represents the estimate of interval between two consecutive false alarm periods, which is composed of the time from correct detection to false alarm and the returning time from false alarms to correct detection. Therefore, mean time between false alarms is the sum of mean time to false alarms and mean returning time from false alarms. Similarly, mean time between missing detections is the sum of mean time to missing detections and mean returning time from missing detections.

In this standard setup, the plant state $x_\mathrm{p}(t)$ and model state $x_\mathrm{m}(t)$ are both incorporated into state dynamics $x(t)$, $w(t)$ represents command input, and $z(t)$ the mismatch error between the plant and desired responses of side slip angle. Under each fixed regime modes $\zeta(t)=i$ and $\eta(t)=j$, the performance measure is selected as the H_∞ norm of closed-loop transfer function from $w(t)$ to $z(t)$, denoted by $\|G_{ij}(s,\Delta)\|_\infty$. It describes the difference between the plant response and the desired one; when $\|G_{ij}(s,\Delta)\|_\infty$ is small, the plant transient behavior of side slip angle is expected to resemble the desired one. The allowable H_∞ bound ρ_i is assumed to be 0.5 for $i=0$ and 0.75 for $i=1$. Hence, when $\zeta(t)=0$, the system is deemed to fail if $\|G_{ij}(s,\Delta)\|_\infty>0.5$ for a duration over hard deadline $T_\mathrm{hd}=5$ s; when $\zeta(t)=1$, it is deemed to fail if $\|G_{ij}(s,\Delta)\|_\infty>0.75$ for a duration over hard deadline. Our design objective is to find a 2DOF controller such that the overall MTTF is greater than 100 s. (Note that frequent incorrect FDI decisions are assumed and this short MTTF design is for demonstration purpose only.)

Figure 2 shows a searching trajectory of Algorithms 1 and 2, where γ_{ij} denotes the probabilistic performance $\Pr\{\|G_{ij}(s,\Delta)\|_\infty\leqslant\rho_i\}$. In the figure, the first plot shows the related probabilistic performance for controller K_0, the second plot for K_1, and the last one shows the trajectory of MTTF when updating controllers iteratively. In the first two plots, the circles represent the expected performance imposed by the gradient search Algorithm 1 in the first design stage, and the triangles represent the achieved probabilistic performance of controllers found by Algorithm 2 in the second stage; in the last plot, the circles represent the MTTF based on expected control performance, and the triangles the achieved MTTF using controllers found in Algorithm 2. As shown in the figure, the achieved probabilistic performance of controllers increases along the direction of expected performance and is greater than that at each iteration. Moreover, MTTF is strictly increasing iteratively, and the following controllers are obtained that achieve MTTF$=511.5348$ s:

$$K_0=\begin{bmatrix}-0.5800 & 0.2251 & -2.1234 & 1.5100 & 4.4991\\ 3.2406 & 0.5472 & 3.8520 & -0.1351 & -6.5038\end{bmatrix},\quad L_0=\begin{bmatrix}1.7530\\ -1.8396\end{bmatrix}$$

$$K_1=\begin{bmatrix}-0.5779 & 0.2122 & -2.1297 & 1.5169 & 4.4946\\ 3.2464 & 0.5368 & 3.8420 & -0.1324 & -6.5095\end{bmatrix},\quad L_1=\begin{bmatrix}1.7533\\ -2.0499\end{bmatrix}$$

To check the transient performance of the closed-loop system, the side slip responses under regime mode $\zeta(t)=0$ and $\eta(t)=0$ for a particular uncertainty sample are shown in Figure 3. It is clear that the plant response has similar transient characteristics as the desired one. As the controller

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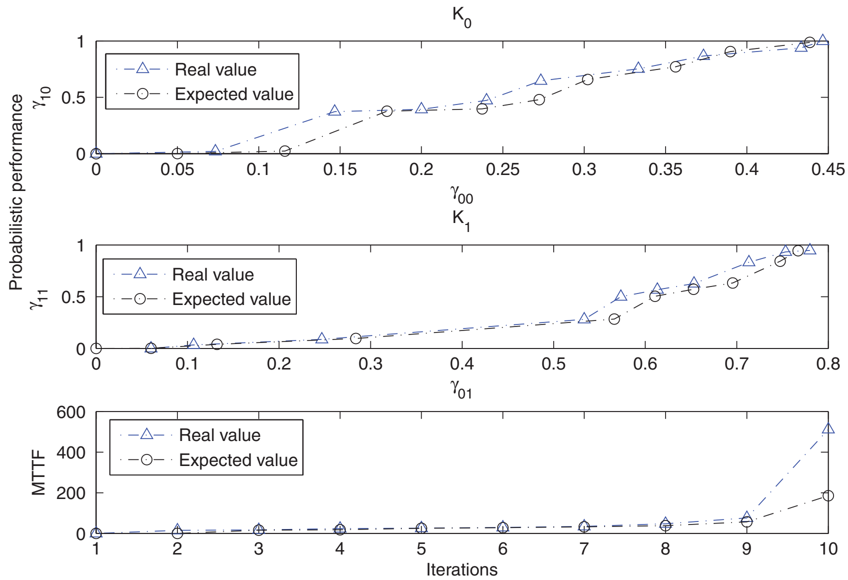

Figure 2. Gradient search trajectory.

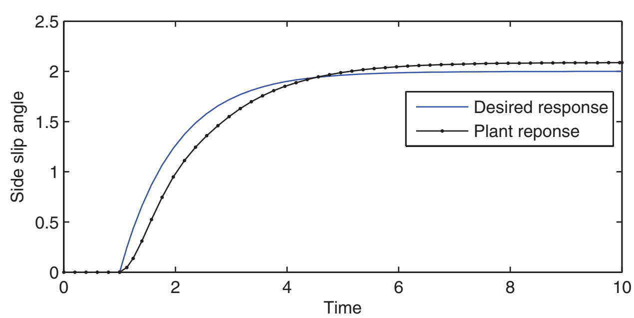

Figure 3. Transient responses in a regime model.

is designed for long-term MTTF and probabilistic modeling uncertainties exist in regime models, there may be differences on static gains for a particular uncertainty sample. Overall, the algorithm provides an effective controller design for MTTF.

8. CONCLUSIONS

This paper discusses the design of MTTF suboptimal controller for FTCSs. The reliability criterion is evaluated from a semi-Markov process model which is built based on probabilistic control performance. But, MTTF cannot be expressed as an analytical expression of controller parameters. Hence, conventional methods are not applicable to controller design with an MTTF objective. To

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overcome this difficulty, a gradient-based search is first carried out on probabilistic performance parameters; the controller is then updated iteratively to achieve this performance. This two-stage method gives a controller achieving the desired MTTF.

APPENDIX A: PROOF OF LEMMA 3

Proof

Recalling (12), because $f(V_{ij})$ is convex in V_{ij}, and V_{ij} is affine in K_j, $v_{ij}(K_j, P_{ij}, \Delta)$ is convex in K_j [26, Chapter 4]. The convexity in P_{ij} will be proved after calculating the gradients.

Let δK_j denote a small perturbation in K_j, and the function value after applying this perturbation is calculated as follows, where V_{ij} denotes the expression in (11) without any perturbations:

$$\begin{aligned} v_{ij}(K_j+\delta K_j, P_{ij}, \Delta) &= f(V_{ij}) + \langle \partial_{V_{ij}} f(V_{ij}), (\delta K_j)^{\mathrm{T}} B_i^{\mathrm{T}} P_{ij} + P_{ij} B_i \delta K_j \rangle + o(\|\delta K_j\|) \\ &= v_{ij}(K_j, P_{ij}, \Delta) + \mathrm{Tr}[\partial_{V_{ij}} f(V_{ij})(\delta K_j)^{\mathrm{T}} B_i^{\mathrm{T}} P_{ij}] \\ &\quad + \mathrm{Tr}[\partial_{V_{ij}} f(V_{ij}) P_{ij} B_i \delta K_j] + o(\|\delta K_j\|) \end{aligned} \tag{A1}$$

Considering that

$$\begin{aligned} \mathrm{Tr}[\partial_{V_{ij}} f(V_{ij})(\delta K_j)^{\mathrm{T}} B_i^{\mathrm{T}} P_{ij}] &= \mathrm{Tr}[(\delta K_j)^{\mathrm{T}} B_i^{\mathrm{T}} P_{ij} \partial_{V_{ij}} f(V_{ij})] \\ &= \mathrm{Tr}[\partial_{V_{ij}} f(V_{ij}) P_{ij} B_i \delta K_j] \end{aligned} \tag{A2}$$

where we have used the facts that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, $\mathrm{Tr}(A) = \mathrm{Tr}(A^{\mathrm{T}})$, and the symmetry of P_{ij} and $\partial_{V_{ij}} f(V_{ij})$. By substituting (A2) into (A1), we have

$$\begin{aligned} v_{ij}(K_j+\delta K_j, P_{ij}, \Delta) &= v_{ij}(K_j, P_{ij}, \Delta) + 2\,\mathrm{Tr}[\partial_{V_{ij}} f(V_{ij}) P_{ij} B_i \delta K_j] + o(\|\delta K_j\|) \\ &= v_{ij}(K_j, P_{ij}, \Delta) + \langle 2 B_i^{\mathrm{T}} P_{ij} \partial_{V_{ij}} f(V_{ij}), \delta K_j \rangle + o(\|\delta K_j\|) \end{aligned}$$

Therefore, $\partial_{K_j} v_{ij}(K_j, P_{ij}, \Delta) = 2B_i^{\mathrm{T}} P_{ij} \partial_{V_{ij}} f(V_{ij}) = 2B_i^{\mathrm{T}} P_{ij} V_{ij}^{+}$. The gradient with respect to P_{ij} can be proved in a similar way as follows:

$$\begin{aligned} v_{ij}(K_j, P_{ij}+\delta P_{ij}, \Delta) &= f(V_{ij}) + \langle \partial_{V_{ij}} f(V_{ij}), A_i^{\mathrm{T}} \delta P_{ij} + \delta P_{ij} A_i + K_j^{\mathrm{T}} B_i^{\mathrm{T}} \delta P_{ij} + \delta P_{ij} B_i K_j \\ &\quad + \delta P_{ij} E_i E_i^{\mathrm{T}} P_{ij}/\rho_i^2 + P_{ij} E_i E_i^{\mathrm{T}} \delta P_{ij}/\rho_i^2 \rangle + o(\|\delta P_{ij}\|) \\ &= v_{ij}(K_j, P_{ij}, \Delta) + \mathrm{Tr}[\partial_{V_{ij}} f(V_{ij})(A_i^{\mathrm{T}} + K_j^{\mathrm{T}} B_i^{\mathrm{T}} + P_{ij} E_i E_i^{\mathrm{T}}/\rho_i^2)\delta P_{ij}] \\ &\quad + \mathrm{Tr}[\partial_{V_{ij}} f(V_{ij}) \delta P_{ij} (A_i + B_i K_j + E_i E_i^{\mathrm{T}} P_{ij}/\rho_i^2)] + o(\|\delta P_{ij}\|) \\ &= v_{ij}(K_j, P_{ij}, \Delta) + \mathrm{Tr}[\partial_{V_{ij}} f(V_{ij})(A_i^{\mathrm{T}} + K_j^{\mathrm{T}} B_i^{\mathrm{T}} + E_i E_i^{\mathrm{T}}/\rho_i^2)\delta P_{ij}] \\ &\quad + \mathrm{Tr}[(A_i + B_i K_j + E_i E_i^{\mathrm{T}}/\rho_i^2)\partial_{V_{ij}} f(V_{ij}) \delta P_{ij}] + o(\|\delta P_{ij}\|) \\ &= v_{ij}(K_j, P_{ij}, \Delta) + \langle (B_i K_j + A_i + E_i E_i^{\mathrm{T}} P_{ij}/\rho_i^2) V_{ij}^{+} \\ &\quad + V_{ij}^{+} (K_j^{\mathrm{T}} B_i^{\mathrm{T}} + A_i^{\mathrm{T}} + P_{ij} E_i E_i^{\mathrm{T}}/\rho_i^2), \delta P_{ij} \rangle + o(\|\delta P_{ij}\|) \end{aligned}$$

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This proves the gradient in P_{ij}. The convexity in P_{ij} can be shown by the following inequality:

$$
\begin{aligned}
&v_{ij}(K_j, P_{ij}+\delta P_{ij}, \Delta)\\
&\geqslant f(V_{ij}) + \langle \partial_{V_{ij}} f(V_{ij}), A_i^{\mathrm{T}}\delta P_{ij} + \delta P_{ij} A_i + K_j^{\mathrm{T}} B_i^{\mathrm{T}} \delta P_{ij} + \delta P_{ij} B_i K_j\\
&\quad + \delta P_{ij} E_i E_i^{\mathrm{T}} P_{ij}/\rho_i^2 + P_{ij} E_i E_i^{\mathrm{T}} \delta P_{ij}/\rho_i^2 + \delta P_{ij} E_i E_i^{\mathrm{T}} \delta P_{ij}/\rho_i^2 \rangle \qquad \text{(A3)}\\
&= v_{ij}(K_j, P_{ij}, \Delta) + \langle \partial_{P_{ij}} v_{ij}(K_j, P_{ij}, \Delta), \delta P_{ij}\rangle + \langle \partial_{V_{ij}} f(V_{ij}), \delta P_{ij} E_i E_i^{\mathrm{T}} \delta P_{ij}/\rho_i^2\rangle \qquad \text{(A4)}\\
&\geqslant v_{ij}(K_j, P_{ij}, \Delta) + \langle \partial_{P_{ij}} v_{ij}(K_j, P_{ij}, \Delta), \delta P_{ij}\rangle \qquad \text{(A5)}
\end{aligned}
$$

Equation (A3) is because of the convexity of f, (A4) follows by substituting $\partial_{P_{ij}} v_{ij}(K_j, P_{ij}, \Delta)$ in (A3), and (A5) is true because that $\mathrm{Tr}[\partial_{V_{ij}} f(V_{ij}) \delta P_{ij} E_i E_i^{\mathrm{T}} \delta P_{ij}/\rho_i^2] \geqslant 0$, resulted from the semi-definite properties of $\partial_{V_{ij}} f(V_{ij})$ and $\delta P_{ij} E_i E_i^{\mathrm{T}} \delta P_{ij}/\rho_i^2$. □

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