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PROBABILISTIC DESIGN OF FAULT TOLERANT CONTROL VIA PARAMETERIZATION*

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Abstract. This paper studies the design of fault tolerant control systems (FTCSs) by considering random faults and two categories of design objectives. The FTCSs are modeled in a stochastic framework, resulting from the random fault process. The design objectives include a stability requirement and a probabilistic performance index. Such an index is chosen as a reliability-related criterion, which can be evaluated from a numerical procedure only but lacks analytical expressions. A parameterization procedure together with a randomization-based optimization method is developed to find a statistically optimal controller that can stabilize the system and achieve the highest reliability.
Key words: Fault tolerant control, reliability, controller parameterization, randomized algorithms.

1. Introduction

Motivated by a growing demand for high reliability and survivability of complex control systems, fault tolerant control (FTC) is gaining increasing consideration from both academia and industry. There are two important components in a typical fault tolerant control system (FTCS): a fault detection and isolation (FDI) scheme and a reconfigurable controller (RC) [1]. Numerous integrated design methodologies developed by considering the interrelationship between FDI and RC have been reported, including the approaches based on adaptive control [30], [37], online fault estimation and control accommodation [36], [38], and robust control FTC [4], which can be collectively categorized as deterministic FTC design approaches.

This paper addresses the design of FTCSs in a different configuration. Consider a plant with a finite set of fault modes S_1, and let $\mathbf{G} = \{G_i : i \in S_1\}$ represent the

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set of dynamical plant models under various fault modes. The evolution of these modes can be represented by a Markov process. Usually the fault mode is not directly known to the controller, and an FDI scheme is used to generate estimates from a finite set S_2. But FDI modes may deviate from true fault modes with an error probability, so another Markov process is adopted to represent FDI modes. The reconfigurable controller $\mathbf{K} = \{K_j : j \in S_2\}$ is in a switching structure, and K_j is engaged for the plant when the FDI is in mode j.

This stochastic FTCS model is preferable to deterministic ones when considering a probabilistic performance criterion. In contrast to the assumption of known regime or fault modes in regular jump linear systems (JLSs), this model assumes unknown fault modes and uses an additional Markov process to represent its estimate, the FDI mode. If the FDI gives a wrong detection mode j, K_j may be used for the plant model G_i, $i \neq j$, even though K_j is originally designed for G_j. As a result of this difference, the design of FTCSs is more challenging, and many existing methods for JLSs cannot be directly applied, e.g., [8], [12], [18], [32]. The related problem in JLSs to this FTC configuration is the partial observation problem [9], which uses conditional probability as the estimation precision of regime modes but cannot describe the estimation delay. In the literature of FTCSs, Mariton studied the effects of this FDI imperfection including detection delay on system stability [23], Srichander and Walker developed the conditions for exponential mean-square stability [28], and much more recent work was also based on this model, such as output feedback stabilization [25], H_2 control [29], and the H_∞ control of a sampled-data system [14]. However, these results considered control objectives only, and the system reliability index was not discussed.

In our problem, in addition to the stability requirement, another design objective $\psi(\mathbf{K})$ of the closed-loop system is evaluated for each controller $\mathbf{K}$ via a numerical method. The design goal is to find an optimal controller $\mathbf{K}^*$ that can optimize $\psi(\mathbf{K})$ subject to the stability constraint. The motivation is to design FTCSs based on reliability-related criteria. Reliability has been deemed the ultimate goal of FTCSs, and many evaluation results have been proposed, e.g., [13], [34], [35]. Most of these methods were based on stochastic modeling, and a new semi-Markov model was developed in [21] considering the dynamical characteristics of FTCSs. Because of the numerical procedures of building and solving stochastic reliability models, reliability criteria cannot be written as analytical functions of $\mathbf{K}$ in general. To overcome this difficulty, stabilizing controller parameterization and randomization-based optimization algorithms are proposed for FTCSs in this paper to find the statistically optimal controller with the highest reliability.

Controller parameterization plays an important role in systems and control theory, which can facilitate the design of optimal controllers by using linear matrix inequalities (LMIs) or other classical optimization techniques. For linear systems, many parameterization results have been reported, such as Youla parameterization [39], H_∞ controller parameterization by Riccati equations and by LMIs [7], [10], [16], covariance controller parameterization [26], [27], and stabilizing controller parameterization by quadratic Lyapunov functions [17]. However, to the best of the authors' knowledge, no controller parameterization result has been reported for FTCSs.

Using LMIs or classical optimization techniques usually requires the parameterization expression and objective function $\psi(\cdot)$ to be affine with respect to free parameters [2]. However, in our problem, even the analytical expression of $\psi(\cdot)$ is not available, and a numerical method is used to calculate $\psi(\cdot)$ [21]. In this case, some statistical methods, such as the randomized algorithms, are useful to perform the design [3], [31], [32].

To recapitulate, this paper presents a parameterization result of stabilizing controllers for stochastic FTCSs and a randomization-based optimization method to search for the statistically optimal controller with respect to a numerical design objective, e.g., a reliability criterion. The remainder of this paper is organized as follows. Section 2 states system model and problem formulation, Section 3 provides some mathematical preliminaries, and Sections 4–7 present the main results: stabilization conditions, controller parameterization, the analysis of stabilizing controller set, and the synthesis of generator matrices. An example is given in Section 8, followed by conclusions in Section 9.

2. Problem formulation

Consider the following Markov dynamical model of FTCSs [23], [28]:

$$\dot{x}(t) = A(\zeta(t))x(t) + B(\zeta(t))u(\eta(t), t), \tag{1}$$

$$y(t) = C(\zeta(t))x(t) + D(\zeta(t))u(\eta(t), t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(\eta(t), t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ denote system state, control input, and output respectively. $A(\zeta(t))$, $B(\zeta(t))$, $C(\zeta(t))$, and $D(\zeta(t))$ are system matrices with appropriate dimensions. (1)–(2) represent a set of linear dynamical models $\mathbf{G} = (G_i : i \in S_1)$, where G_i denotes the dynamical model when $\zeta(t) = i$.

$\zeta(t)$ and $\eta(t)$ are assumed to be two separate continuous-time Markov processes with finite state spaces $S_1 = \{0, 1, 2, \ldots, N_1\}$ and $S_2 = \{0, 1, 2, \ldots, N_2\}$ to represent system faults and FDI results respectively. $\zeta(t)$ is a homogeneous process, whereas the transition rates of $\eta(t)$ depend on the current state of $\zeta(t)$.

The transition probability of the fault process $\zeta(t)$ from mode i to j, $i, j \in S_1$, in the infinitesimal time interval of $\triangle t$, is given by

$$\zeta(t) : p_{ij}(\triangle t) = \begin{cases} \alpha_{ij}\triangle t + o(\triangle t), & i \neq j, \\ 1 + \alpha_{ii}\triangle t + o(\triangle t), & i = j, \end{cases}$$

where $\alpha_{ij} \geq 0$ and $\alpha_{ii} = -\sum_{j\in S_1, j\neq i} \alpha_{ij}$ denote the transition rates of $\zeta(t)$, and $o(\triangle t)$ denotes higher-order infinitesimal terms. When $\zeta(t) = k$, $k \in S_1$, the

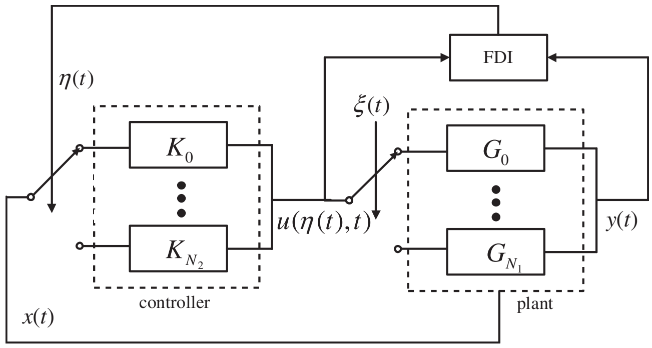

Figure 1. The system structure.

transition probability of $\eta(t)$ from mode i to j, $i, j \in S_2$, in the infinitesimal time interval of $\triangle t$, is given by

$$\eta(t) : p_{ij}^k(\triangle t) = \begin{cases} \beta_{ij}^k \triangle t + o(\triangle t), & i \neq j, \\ 1 + \beta_{ii}^k \triangle t + o(\triangle t), & i = j, \end{cases}$$

where $\beta_{ij}^k \geq 0$ and $\beta_{ii}^k = -\sum_{j \in S_2, j \neq i} \beta_{ij}^k$ represent the transition rates of $\eta(t)$ given $\zeta(t) = k$. In general, these transition rates, α_{ij} and β_{ij}^k, compose the generator matrices of $\zeta(t)$ and $\eta(t)$ respectively, denoted by $F = [\alpha_{ij}]_{N_1 \times N_1}$ and $H^k = [\beta_{ij}^k]_{N_2 \times N_2}$.

The closed-loop system structure is shown in Figure 1. Consider the static state-feedback controller, $u(\eta(t), t) = K(\eta(t))x(t)$. For simplicity, we write $u_j(t) = K_j x(t)$ for $\eta(t) = j \in S_2$. The controller is composed of a set of static gains, denoted by $\mathbf{K} = \{K_0, K_1, \ldots, K_{N_2}\}$. When $\eta(t)$ indicates the fault mode i, K_i is in use. In practice, it is impossible to have a "perfect" FDI that always instantaneously indicates the correct fault mode. Hence, there may be a mismatch between $\eta(t)$ and $\zeta(t)$. In this case, finding $\mathbf{K}$ to achieve closed-loop stability is our first concern in the design of FTCSs.

Remark 1. The interaction between $\zeta(t)$ and $\eta(t)$ causes the major difficulty in the stabilizing design of FTCSs. This is the main difference between FTCSs and regular JLSs.

Such a stabilizing $\mathbf{K}$ is usually not unique. In fact, the set of all stabilizing $\mathbf{K}$ can be found via parameterization. When considering a more specific performance criterion $\psi(\mathbf{K})$, it is desirable to obtain the optimal stabilizing controller $\mathbf{K}^*$ with respect to $\psi(\mathbf{K})$. This leads to the second stage of design. In this paper, such a $\psi(\mathbf{K})$ is chosen as a reliability criterion. To reflect the dynamical characteristics of FTCSs, the following reliability function $R_{\text{lb}}(t)$ was defined in [21].

Definition 1. The reliability function $R_{\text{lb}}(t)$ of FTCSs is defined as the probability that, during time interval $[0, t]$, FTCSs either satisfy presumed control objectives or violate them only temporally for a short time no longer than the presumed hard deadline.

$R_{\rm lb}(t)$ is a function criterion, and an alternative scalar reliability criterion, mean time to failure (MTTF), is often preferable in controller design. MTTF is defined as the expectation of satisfactory operation time, i.e., the expected lifetime of overall FTCSs, and is equal to

$$\text{MTTF} = \int_0^\infty R_{\rm lb}(t)\mathrm{d}t. \tag{3}$$

A stochastic process was built in [21] to describe the evolution of control performance under fault occurrences and controller reconfigurations. $R_{\rm lb}(t)$ and MTTF can be calculated based on the transition and stationary probabilities of the stochastic process. However, neither of these two reliability criteria has analytic function expressions available. In this paper, $\psi(\mathbf{K})$ is selected as the scalar reliability index, MTTF.

Based on such a $\psi(\mathbf{K})$, a randomization procedure is available to find a statistical optimum $\hat{\mathbf{K}}^*$, an estimate of $\mathbf{K}^*$, such that

$$\Pr\{\psi(\mathbf{K}^*) - \psi(\hat{\mathbf{K}}^*) > \epsilon\} \leq \delta, \tag{4}$$

where $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$ are precision parameters of the estimate.

The main procedure of the randomized algorithm presented in [31] is summarized as follows, where the key step is to find a parameterization set of stabilizing controllers: $\mathcal{K} \triangleq \{\text{all stabilizing } \mathbf{K}\} = \{\mathbf{K}|\mathbf{K} = \varphi(z),\ z \in \Omega\}$, where $\varphi : \Omega \to \mathcal{K}$ denotes the parameterization mapping from a free parameter z within a bounded set Ω to a stabilizing controller $\mathbf{K}$.

Algorithm 1. Estimate the statistical optimum

(1) Determine a suitable sample quantity M_1 based on the precision parameters ϵ and δ and statistical theory, e.g., the Chernoff bound [3, p. 123].
(2) Generate M_1 independent samples $z^{(1)}, \ldots, z^{(M_1)}$ in Ω according to the distribution of z. Calculate the corresponding controllers $\mathbf{K}^{(i)} = \varphi(z^{(i)})$, $i = 1, \ldots, M_1$.
(3) Evaluate the performance value at each sample controller $\mathbf{K}^{(i)}$:

$$\psi_i = \psi(\mathbf{K}^{(i)}) = \psi(\varphi(z^{(i)})), \qquad i = 1, \ldots, M_1.$$

Let z_0 denote the parameter such that $\psi(\varphi(z_0)) = \max_{1\leq i\leq M_1} \psi_i$. Then $\hat{\mathbf{K}}^* = \varphi(z_0)$.

The remainder is then focused on developing a parameterization method for Algorithm 1.

3. Preliminaries

The following notation is used throughout the paper. A^{-T} means $(A^T)^{-1}$. $A^{\perp}$ denotes a matrix with the following properties: $\mathcal{N}(A^{\perp}) = \mathcal{R}(A)$ and $A^{\perp}A^{\perp T} > 0$, where $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null and range spaces of A respectively. $\triangleq$ is used for notation definitions. $\|\cdot\|$ denotes the Euclidean norm for vectors and the largest singular value for matrices. $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ the set of nonnegative integers. For notational simplicity, in (1)–(2), for $\zeta(t) = i,\ \eta(t) = j,\ i \in S_1,\ j \in S_2$, denote $A_i \triangleq A(\zeta(t)),\ B_i \triangleq B(\zeta(t)),\ C_i \triangleq C(\zeta(t)),\ D_i \triangleq D(\zeta(t))$, and $u_j(t) \triangleq u(\eta(t), t)$.

Definition 2 (EMS stability [28])**.** An FTCS is said to be exponentially mean-square (EMS) stable if for any initial Markov states at $t = 0$, $\zeta(0)$, and $\eta(0)$, there exist $a > 0$, $b > 0$, and some number $\delta(\zeta(0), \eta(0)) > 0$, such that when $\|x(0)\| \le \delta(\zeta(0), \eta(0))$, the following inequality holds for $t \ge 0$:

$$E\{\|x(t)\|^2\} \le b\|x(0)\|^2 e^{-at},$$

where $E\{\cdot\}$ denotes the mathematical expectation.

Lemma 1 (Stability conditions [28])**.** *An FTCS in (1)–(2) is stabilized in the sense of EMS stability by the static state-feedback control law*

$$u_i(t) = K_i x(t), \qquad i \in S_2,$$

if and only if for any given $k \in S_1$ and $i \in S_2$ there exist positive definite matrices $P_{ik} > 0$, satisfying

$$\tilde{A}_{ik}^T P_{ik} + P_{ik}\tilde{A}_{ik} + \sum_{j \in S_2, j \ne i} \beta_{ij}^k P_{jk} + \sum_{j \in S_1, j \ne k} \alpha_{kj} P_{ij} < 0,$$

where

$$\tilde{A}_{ik} \triangleq A_k + B_k K_i - 0.5 \sum_{j \in S_2, j \ne i} \beta_{ij}^k - 0.5 \sum_{j \in S_1, j \ne k} \alpha_{kj}.$$

Lemma 1 can be used for stability analysis for a given state-feedback controller, but it is difficult to solve K_i directly using these inequalities. The main difficulty lies in the fact that the number of gains K_i is less than that of the inequalities involved in the above condition such that each K_i must satisfy multiple inequalities simultaneously. In contrast, regular JLSs do not have this problem, and the controller can be solved using LMIs [12]. The partial observation problem of the JLS considered in [33] has a similar form as in FTCSs but only a sufficient condition was derived. See [22], [29] for more discussions.

The following two lemmas are introduced for the purpose of deriving stabilization conditions and a parameterization set.

Lemma 2 (Finsler's theorem [16], [27])**.** *Let matrices $M \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{n\times n}$ be given, and assume that* $\operatorname{rank}(M) < n$ *and $Q = Q^T$. Let (M_L, M_R) be* *any full rank factors of* M *such that* $M = M_L M_R$ *and* $\mathrm{rank}(M_L) = \mathrm{rank}(M_R) = \mathrm{rank}(M)$. *Then*

$$M^{\perp} Q M^{\perp T} < 0$$

if and only if

$$\mu M M^T - Q > 0$$

for some $\mu \in \mathbb{R}$. *If the above condition holds, all such* μ *are given by*

$$\mu > \mu_{\min} \triangleq \lambda_{\max}[N(Q - QM^{\perp T}(M^{\perp} Q M^{\perp T})^{-1} M^{\perp} Q)N^T],$$

where $\lambda_{\max}(\cdot)$ *denotes the largest eigenvalue, and* $N \triangleq (M_R M_R^T)^{(-1/2)}$.

Lemma 3 (Projection lemma and parameterization set)**.** *Let matrices* $M \in \mathbb{R}^{n\times m}$ *and* $Q = Q^T \in \mathbb{R}^{n\times n}$ *be given. The following two statements are equivalent.*

(1) There exists a matrix X *satisfying*

$$MX + (MX)^T + Q < 0. \tag{5}$$

(2) The following condition holds:

$$M^{\perp} Q M^{\perp T} < 0 \ \text{ or } \ M M^T > 0. \tag{6}$$

If statement 2 holds, all matrices X *satisfying statement 1 are given by*

$$X = g(L, \rho|M, Q) \triangleq -\rho^{-1} M^T + \rho^{-1/2} L(\rho^{-1} M M^T - Q)^{1/2}, \tag{7}$$

where L *is an arbitrary matrix satisfying* $\|L\| < 1$, *and* $\rho \in (0, \ \rho_{\max})$ *a positive scalar.* L *and* ρ *are immediate variables of function* g, *and the symbol* $|$ *in (7) is used to indicate the dependence of* X *on* M *and* Q.

$\rho_{\max} = a^{-1}$ *is calculated by solving the following LMI problem:*

$$\min_{\{a,X\}} a$$

subject to

$$a > 0,$$
$$\begin{bmatrix} -aI & X \\ X^T & MX + (MX)^T + Q \end{bmatrix} < 0. \tag{8}$$

Moreover, $\rho \in (0, \ \rho_{\max})$ *if and only if it satisfies*

$$\rho^{-1} M M^T - Q > 0, \tag{9}$$

which ensures that $(\rho^{-1} M M^T - Q)^{1/2}$ *exists and that (7) is valid.*

Lemma 3 is adopted from Corollary 2.3.9 in [27] with modifications to make it suitable for our problem. The proof is given in the Appendix. For a given inequality in the form of (5), Lemma 3 provides a solvability condition and a parameterization set of all its solutions:

$$\mathcal{G}_{M,Q} = \{X | X = g(L, \rho|M, Q), \ \|L\| < 1, \ \rho \in (0, \rho_{\max})\}, \tag{10}$$

where $g(L, \rho|M, Q)$ is defined in (7).

4. Stabilization conditions

Let us begin with the case that the state spaces of $\zeta(t)$ and $\eta(t)$, S_1 and S_2, are both equal to $\{0, 1\}$, where 0 denotes the fault-free situation and 1 the faulty mode. This type of FTCS is referred to as the basic case in what follows. The stochastic behavior of $\zeta(t)$ is governed by its generator matrix F; when $\zeta(t) = 0$ or 1, the behavior of $\eta(t)$ is determined by the corresponding generator matrix H^0 or H^1 [5], [20].

The generator matrices are composed of the transition rates of $\zeta(t)$ and $\eta(t)$, α_{ij} and β^k_{ij}, which have the following forms for the basic case:

$$F = \begin{bmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{bmatrix}, \qquad H^0 = \begin{bmatrix} \beta^0_{00} & \beta^0_{01} \\ \beta^0_{10} & \beta^0_{11} \end{bmatrix}, \qquad H^1 = \begin{bmatrix} \beta^1_{00} & \beta^1_{01} \\ \beta^1_{10} & \beta^1_{11} \end{bmatrix}.$$

For the system in (1)–(2), by Lemma 1, $\{K_0,\ K_1\}$ stabilizes the FTCS in the sense of EMS stability if and only if there exist positive definite matrices P_{ik}, $i \in S_2$, $k \in S_1$, such that the following inequalities hold simultaneously:

$$P_{00}B_0K_0 + (P_{00}B_0K_0)^T + Q_{00} < 0, \tag{11}$$

$$P_{10}B_0K_1 + (P_{10}B_0K_1)^T + Q_{10} < 0, \tag{12}$$

$$P_{01}B_1K_0 + (P_{01}B_1K_0)^T + Q_{01} < 0, \tag{13}$$

$$P_{11}B_1K_1 + (P_{11}B_1K_1)^T + Q_{11} < 0, \tag{14}$$

where Q_{ik}, $i \in S_2$, $k \in S_1$, is defined as

$$\begin{aligned} Q_{ik} \triangleq {} & (A_k - 0.5\beta^k_{i(1-i)} - 0.5\alpha_{k(1-k)})^T P_{ik} + P_{ik}(A_k - 0.5\beta^k_{i(1-i)} - 0.5\alpha_{k(1-k)}) \\ & + \beta^k_{(1-i)k} P_{i(1-k)} + \alpha_{k(1-k)} P_{i(1-k)}. \end{aligned} \tag{15}$$

The set of all stabilizing controllers can be captured naturally by posing a matrix inequality problem (11)–(14) for $\{K_0, K_1\}$. Note that both K_0 and K_1 appear in two inequalities. So the intersection of the solution sets of (11) and (13) gives the set of K_0, and K_1 can be obtained in a similar way from (12) and (14).

Lemma 4. *For the basic case of an FTCS in (1)–(2), if B_0 and B_1 are row rank deficient, then there exists a stabilizing state-feedback controller $\{K_0,\ K_1\}$ in the sense of EMS stability only if there exist positive definite matrices P_{ik}, $k \in S_1 = \{0, 1\}$, $i \in S_2 = \{0, 1\}$, such that*

$$(P_{00}B_0)^\perp Q_{00} (P_{00}B_0)^{\perp T} < 0, \tag{16}$$

$$(P_{10}B_0)^\perp Q_{10} (P_{10}B_0)^{\perp T} < 0, \tag{17}$$

$$(P_{01}B_1)^\perp Q_{01} (P_{01}B_1)^{\perp T} < 0, \tag{18}$$

$$(P_{11}B_1)^\perp Q_{11} (P_{11}B_1)^{\perp T} < 0, \tag{19}$$

where Q_{ik} is defined in (15). If B_0 has full row rank, (16) and (17) are removed from the conditions; if B_1 has full row rank, (18) and (19) are removed.

This lemma is derived based on Lemma 1, and the proof is given in the Appendix. By converting the inequalities in Lemma 4 to LMIs, we have the following theorem.

Theorem 1. *For the basic case of FTCS in (1)–(2), if B_0 and B_1 are row rank deficient, and all the transition rates of $\zeta(t)$ and $\eta(t)$ are nonzero, then there exist stabilizing state-feedback controllers in the sense of EMS stability only if there exist positive definite matrices P_{ik}, positive scalars μ_{ik}, $k \in S_1 = \{0, 1\}$, $i \in S_2 = \{0, 1\}$, such that*

$$\begin{bmatrix} P_{00}^{-1}\bar{A}_{00}^T + \bar{A}_{00}P_{00}^{-1} - \mu_{00}B_0B_0^T & P_{00}^{-1} & P_{00}^{-1} \\ P_{00}^{-1} & -P_{10}^{-1}/\beta_{01}^0 & 0 \\ P_{00}^{-1} & 0 & -P_{01}^{-1}/\alpha_{01} \end{bmatrix} < 0, \quad (20)$$

$$\begin{bmatrix} P_{10}^{-1}\bar{A}_{10}^T + \bar{A}_{10}P_{10}^{-1} - \mu_{10}B_0B_0^T & P_{10}^{-1} & P_{10}^{-1} \\ P_{10}^{-1} & -P_{00}^{-1}/\beta_{10}^0 & 0 \\ P_{10}^{-1} & 0 & -P_{11}^{-1}/\alpha_{01} \end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} P_{01}^{-1}\bar{A}_{01}^T + \bar{A}_{01}P_{01}^{-1} - \mu_{01}B_1B_1^T & P_{01}^{-1} & P_{01}^{-1} \\ P_{01}^{-1} & -P_{11}^{-1}/\beta_{01}^1 & 0 \\ P_{01}^{-1} & 0 & -P_{00}^{-1}/\alpha_{10} \end{bmatrix} < 0, \quad (22)$$

$$\begin{bmatrix} P_{11}^{-1}\bar{A}_{11}^T + \bar{A}_{11}P_{11}^{-1} - \mu_{11}B_1B_1^T & P_{11}^{-1} & P_{11}^{-1} \\ P_{11}^{-1} & -P_{01}^{-1}/\beta_{10}^1 & 0 \\ P_{11}^{-1} & 0 & -P_{10}^{-1}/\alpha_{10} \end{bmatrix} < 0, \quad (23)$$

where $\bar{A}_{ik} \triangleq A_k - 0.5\beta_{i(1-i)}^k - 0.5\alpha_{k(1-k)}$. If B_0 has full row rank, (20) and (21) are removed from the conditions; if B_1 has full row rank, (22) and (23) are removed. If some transition rates are zero, the corresponding rows and columns containing those zero transition rates are removed from the above matrices.

Proof. Take (16) as an example, and the derivations are similar for the other three inequalities. As $P_{00} > 0$ and $(P_{00}B_0)^{\perp}((P_{00}B_0)^{\perp})^T > 0$, both $(P_{00}B_0)^{\perp}$ and $(P_{00}B_0)^{\perp}P_{00}$ have full row rank. Considering $(P_{00}B_0)^{\perp}P_{00}B_0 = 0$ and $(P_{00}B_0)^{\perp}P_{00} = B_0^{\perp}$, we have

$$(P_{00}B_0)^{\perp} = B_0^{\perp}P_{00}^{-1}.$$

So (16) is equivalent to

$$B_0^{\perp}P_{00}^{-1}Q_{00}P_{00}^{-1}B_0^{\perp T} < 0.$$

Substitute Q_{00} and denote $\bar{A}_{00} \triangleq A_0 - 0.5\beta_{01}^0 - 0.5\alpha_{01}$ to obtain

$$B_0^{\perp}(P_{00}^{-1}\bar{A}_{00}^T + \bar{A}_{00}P_{00}^{-1} + \beta_{01}^0P_{00}^{-1}P_{10}P_{00}^{-1} + \alpha_{01}P_{00}^{-1}P_{01}P_{00}^{-1})B_0^{\perp T} < 0.$$

By Lemma 2, this inequality is equivalent to

$$P_{00}^{-1}\bar{A}_{00}^{T} + \bar{A}_{00}P_{00}^{-1} + \beta_{01}^{0}P_{00}^{-1}P_{10}P_{00}^{-1} + \alpha_{01}P_{00}^{-1}P_{01}P_{00}^{-1} < \mu_{00}B_0B_0^T, \quad (24)$$

where $\mu_{00} \in \mathbb{R}$. Pre- and post-multiply P_{00},

$$\bar{A}_{00}^{T}P_{00} + P_{00}\bar{A}_{00} + \beta_{01}^{0}P_{10} + \alpha_{01}P_{01} < \mu_{00}P_{00}B_0B_0^T P_{00}. \quad (25)$$

According to Lemma 2, all feasible μ_{00} are given by $\mu_{00} > \mu_{00\min}$, where $\mu_{00\min}$ can be calculated by the parameters in the inequality. Therefore, if the feasible set of μ_{00} is nonempty, there must be a feasible $\mu_{00} > 0$. Furthermore, we need to consider only the positive case of μ_{00} to obtain all the feasible P_{ij}, by the following reasoning.

Suppose for any two feasible values of μ_{00}, $\mu_1 \leq 0$ and $\mu_2 > 0$, all the corresponding feasible solutions of P_{ij} in (24), $i, j \in \{0, 1\}$, are denoted by $\mathcal{P}_1$ and $\mathcal{P}_2$. For every element $P_{ij} \in \mathcal{P}_1$, $i, j \in \{0, 1\}$, (24) holds for this P_{ij} and μ_1. Again, based on Lemma 2, this element P_{ij}, $i, j \in \{0, 1\}$, is also feasible for (24) corresponding to μ_2 as $\mu_2 > \mu_1$ and thereby belongs to $\mathcal{P}_2$. Therefore, $\mathcal{P}_1 \subseteq \mathcal{P}_2$, which means that the feasible solution of P_{ij}, $i, j \in \{0, 1\}$, for (24) when $\mu \leq 0$ is a subset of those when $\mu > 0$, and we need to consider this positive case only.

Suppose that the transition rates $\beta_{01}^{0} > 0$ and $\alpha_{01} > 0$. By the Schur's complement lemma [2], (24) is equivalent to

$$\begin{bmatrix} P_{00}^{-1}\bar{A}_{00}^{T} + \bar{A}_{00}P_{00}^{-1} - \mu_{00}B_0B_0^T & P_{00}^{-1} & P_{00}^{-1} \\ P_{00}^{-1} & -P_{10}^{-1}/\beta_{01}^{0} & 0 \\ P_{00}^{-1} & 0 & -P_{01}^{-1}/\alpha_{01} \end{bmatrix} < 0. \quad (26)$$

If some transition rate is zero, the corresponding term involving zero rate in (24) is removed, and so are the corresponding row and column in (26). For example, if $\alpha_{01} = 0$, (26) becomes

$$\begin{bmatrix} P_{00}^{-1}\bar{A}_{00}^{T} + \bar{A}_{00}P_{00}^{-1} - \mu_{00}B_0B_0^T & P_{00}^{-1} \\ P_{00}^{-1} & -P_{10}^{-1}/\beta_{01}^{0} \end{bmatrix} < 0.$$

Similarly, (17)–(19) can also be converted to LMI's that are affine in P_{00}^{-1}, P_{01}^{-1}, P_{10}^{-1}, P_{11}^{-1}, μ_{00}, μ_{01}, μ_{10}, and μ_{11}. □

Remark 2. The above results are for the basic case of FTCSs, and can be readily modified for the cases of multiple fault modes. For example, if $S_1 = S_2 = \{0, 1, 2\}$, to ensure stochastic stability, there are 9 inequalities in Theorem 1, and a typical one is

$$\begin{bmatrix} \begin{matrix} P_{00}^{-1}\bar{A}_{00}^{T}P_{00}P_{00}^{-1}\bar{A}_{00} \\ \cdot P_{00}^{-1} - \mu_{00}B_0B_0^T \end{matrix} & P_{00}^{-1} & P_{00}^{-1} & P_{00}^{-1} & P_{00}^{-1} \\ P_{00}^{-1} & -P_{10}^{-1}/\beta_{01}^{0} & 0 & 0 & 0 \\ P_{00}^{-1} & 0 & -P_{20}^{-1}/\beta_{02}^{0} & 0 & 0 \\ P_{00}^{-1} & 0 & 0 & P_{01}^{-1}/\alpha_{01} & 0 \\ P_{00}^{-1} & 0 & 0 & 0 & P_{02}^{-1}/\alpha_{02} \end{bmatrix} < 0.$$

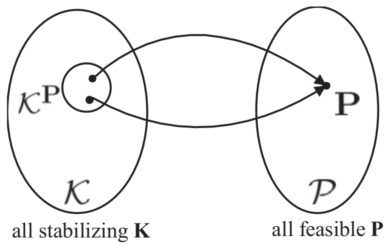

Figure 2. Relationship between $\mathcal{P}$ and $\mathcal{K}$.

Theorem 1 gives conditions on P_{ij}, $i, j \in \{0, 1\}$, to ensure that each single inequality in (16)–(19) has feasible solutions. The stabilizing controller $\mathbf{K} = \{K_0, K_1\}$ satisfying these four inequalities simultaneously can be generated by a randomization procedure presented in the next section.

5. Controller parameterization

Recall Lemma 3 and (10), and denote

$$\begin{aligned} &\mathcal{K}^{\mathbf{P}} \triangleq \{\{K_0, K_1\} | K_0 \in \mathcal{W}_{00} \cap \mathcal{W}_{01},\ K_1 \in \mathcal{W}_{10} \cap \mathcal{W}_{11}, \\ &\mathcal{W}_{ij} \triangleq \mathcal{G}_{P_{ij}B_j, Q_{ij}},\ i, j \in \{0, 1\}\}, \end{aligned} \tag{27}$$

where $\mathbf{P} \triangleq \{P_{ij},\ i, j \in \{0, 1\}\}$. So $\mathcal{K}^{\mathbf{P}}$ is the set of stabilizing controllers associated with $\mathbf{P}$. Let $\mathcal{P} \triangleq \{\mathbf{P} | \mathbf{P}$ satisfies Theorem 1$\}$, the set of all $\mathbf{P}$ satisfying Theorem 1. $\mathbf{P} \in \mathcal{P}$ ensures that $\mathcal{W}_{ij} \neq \emptyset$, where $\emptyset$ denotes the empty set. The set of all stabilizing controllers is denoted as

$$\mathcal{K} \triangleq \{\text{all stabilizing } \mathbf{K}\} = \bigcup_{\mathbf{P} \in \mathcal{P}} \mathcal{K}^{\mathbf{P}}. \tag{28}$$

Figure 2 illustrates the relationship between $\mathcal{P}$ and $\mathcal{K}$: Each $\mathbf{K} \in \mathcal{K}$ corresponds to some $\mathbf{P} \in \mathcal{P}$; if $\mathcal{K}^{\mathbf{P}} \neq \emptyset$, all its elements correspond to and can be generated by $\mathbf{P}$ using a randomization procedure; if $\mathcal{K}^{\mathbf{P}} = \emptyset$, find another $\mathbf{P} \in \mathcal{P}$, and repeat the procedure.

The problem considered in this section is to check whether $\mathcal{K}^{\mathbf{P}} = \emptyset$ or not given $\mathbf{P} \in \mathcal{P}$; furthermore, if $\mathcal{K}^{\mathbf{P}} \neq \emptyset$, generate samples in $\mathcal{K}^{\mathbf{P}}$.

Based on (27), denote $\mathcal{K}_0^{\mathbf{P}} \triangleq \mathcal{W}_{00} \cap \mathcal{W}_{01}$ and $\mathcal{K}_1^{\mathbf{P}} \triangleq \mathcal{W}_{10} \cap \mathcal{W}_{11}$. Then $\mathcal{K}^{\mathbf{P}} = \mathcal{K}_0^{\mathbf{P}} \times \mathcal{K}_1^{\mathbf{P}}$, where $\times$ denotes the Cartesian product. So $\mathcal{K}^{\mathbf{P}} \neq \emptyset$ if and only if $\mathcal{K}_0^{\mathbf{P}} \neq \emptyset$ and $\mathcal{K}_1^{\mathbf{P}} \neq \emptyset$. Take $\mathcal{K}_0^{\mathbf{P}}$ as an example for the following derivation, and the same procedure follows for $\mathcal{K}_1^{\mathbf{P}}$.

As shown in Figure 3, the basic idea is to generate samples in $\mathcal{W}_{00} = \mathcal{G}_{P_{00}B_0, Q_{00}}$ and to test condition (13) for $\mathcal{W}_{01}$ to obtain $K_0 \in \mathcal{K}_0^{\mathbf{P}}$. Recall (10) and (27), and let the free parameters L and ρ be uniformly distributed random

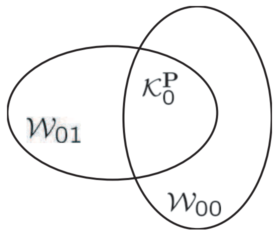

Figure 3. Illustration of controller generation.

variables. $K_0 = g(L, \rho | P_{00} B_0, Q_{00}) \in \mathcal{W}_{00}$ can be generated by L and ρ, where $g(\cdot, \cdot|\cdot, \cdot)$ is defined in (7). Obviously, $\mathcal{K}_0^{\mathbf{P}} \neq \emptyset$ if and only if the following probability is nonzero:

$$\Pr\{K_0 \in \mathcal{K}_0^{\mathbf{P}} | K_0 \in \mathcal{W}_{00}\} = \Pr\{K_0 \text{ satisfies (13)} | K_0 \in \mathcal{W}_{00}\}. \tag{29}$$

Define an indicator function

$$I(L, \rho) = \begin{cases} 1, & K_0 \in \mathcal{K}_0^{\mathbf{P}} \text{ given } K_0 = g(L, \rho | P_{00} B_0, Q_{00}) \in \mathcal{W}_{00}; \\ 0, & \text{otherwise,} \end{cases}$$

and then $\Pr\{I(L, \rho) = 1\} = \Pr\{K_0 \in \mathcal{K}_0^{\mathbf{P}} | K_0 \in \mathcal{W}_{00}\}$. According to Chernoff's bound [3, p. 123], when generating $N \geq \frac{\ln(2/\delta_2)}{2\epsilon_2^2}$ identically and independently distributed (i.i.d.) samples for $\delta_2 > 0$ and $\epsilon_2 > 0$, the following statistic provides an estimate of the probability in (29):

$$\hat{P}_N = \frac{\sum_{i=1}^{N} I(L_i, \rho_i)}{N}, \tag{30}$$

where L_i and ρ_i denote i.i.d. samples of L and ρ respectively. Furthermore, it satisfies

$$\Pr\{|\Pr\{I(L, \rho) = 1\} - \hat{P}_N| \leq \epsilon_2\} \geq 1 - \delta_2. \tag{31}$$

Suppose that ϵ_2 and δ_2 are so small that we can use the estimate $\hat{P}_N$ as the true probability in (29). So $\mathcal{K}_0^{\mathbf{P}} \neq \emptyset$ is equivalent to $\hat{P}_N > 0$, which solves the first problem of this section.

If $\mathcal{K}_0^{\mathbf{P}} \neq \emptyset$, we can then generate elements in $\mathcal{W}_{00}$ and test (13) to obtain samples in $\mathcal{K}_0^{\mathbf{P}}$. Recall Algorithm 1 in Section 2, and suppose M_1 stabilizing controllers are needed. The next problem is to determine the number of $K_0 \in \mathcal{W}_{00}$ to be tested in order to generate M_1 controllers $K_0 \in \mathcal{K}_0^{\mathbf{P}}$.

For M_2 i.i.d. samples L_i and ρ_i, denote $Y_i = I(L_i, \rho_i)$, $i = 1, \ldots, M_2$. So $\sum_{i=1}^{M_2} Y_i$ is the number of $K_0 \in \mathcal{K}_0^{\mathbf{P}}$ and subject to the following binomial distribution:

$$\Pr\left\{\sum_{i=1}^{M_2} Y_i \geq M_1\right\} = \sum_{k=M_1}^{M_2} \binom{M_2}{k} (\hat{P}_N)^k (1 - \hat{P}_N)^{M_2 - k}. \tag{32}$$

Set a confidence level δ_3, and select M_2 to ensure $\Pr\{\sum_{i=1}^{M_2} Y_i \geq M_1\} \geq 1 - \delta_3$. This means that when testing M_2 samples in $\mathcal{W}_{00}$, M_1 samples of $K_0^i \in \mathcal{K}_0^{\mathbf{P}}$ are obtained with probability $1 - \delta_3$. The procedures of generating M_1 controllers are summarized in Algorithm 2.

Algorithm 2. Controller generation

(1) Let $i = 0$.
(2) For K_i, estimate $\Pr\{K_i \in \mathcal{K}_i^{\mathbf{P}} | K_i \in \mathcal{W}_{i0}\}$ by $\hat{P}_N$ in (30) for some small parameters ϵ_2 and δ_2. If $\hat{P}_N = 0$, no stabilizing controller exists and stop.
(3) For a small confidence level δ_3, select M_2 such that

$$\sum_{k=M_1}^{M_2} \binom{M_2}{k} (\hat{P}_N)^k (1 - \hat{P}_N)^{M_2-k} \geq 1 - \delta_3.$$

(4) Generate M_2 samples in set $\mathcal{W}_{i0} = \mathcal{G}_{P_{i0}B_0, Q_{i0}}$. Test (13) if $i = 0$ or (14) if $i = 1$ for each sample, and record those stabilizing controllers in $\mathcal{K}_i^{\mathbf{P}}$.
(5) Let $i = 1$, and follow steps 2 through 4 to generate the controller samples for K_1.

Remark 3. Algorithm 2 still applies when there are multiple fault modes. For example, if $S_1 = S_2 = \{0, 1, 2\}$, there are 9 similar inequalities in Theorem 1, and each controller $\{K_0, K_1, K_2\}$ is in the intersection of three solution sets.

Algorithm 2 generates the controllers for step 2 of Algorithm 1 in Section 2. The design procedure of FTCSs is finally established as follows by combining Algorithms 1 and 2.

Design procedure

(1) Determine a suitable sample quantity M_1 based on the precision parameters ϵ and δ.
(2) Solve (20)–(23) in Theorem 1 for $\mathbf{P}$.
(3) Use Algorithm 2 to generate M_1 stabilizing controllers corresponding to $\mathbf{P}$.
(4) If M_1 controllers in $\mathcal{K}^{\mathbf{P}}$ are successfully generated, follow step 3 in Algorithm 1 on the generated controllers, and find the statistical optimum $\hat{\mathbf{K}}^*$. If $\mathcal{K}^{\mathbf{P}} = \emptyset$, go to step 2 to solve for an alternative $\mathbf{P}$.

If this procedure fails to find nonempty $\mathcal{K}^{\mathbf{P}}$, the system is said to be not stabilizable.

Remark 4. Note that the freedom of $\mathbf{P}$ in (28) is not exploited in this design procedure, although it is possible to obtain a set of feasible solutions $\mathbf{P}$ satisfying Theorem 1 by varying the settings in the LMI solver: the target value for the auxiliary convex program of the feasibility problem [11]. But this may lead to controllers with larger magnitudes, which is not preferable in practice due to excessive control energy. So we do not solve a set of $\mathbf{P}$ and optimize among controllers with different orders of magnitudes.

Remark 5. The parameterization method can be extended to static output-feedback controllers $u(\eta(t), t) = K(\eta(t))y(t)$, provided that $D(\zeta(t)) = 0$ in (2). Using output-feedback controllers for this special case is equivalent to replacing $K(\eta(t))$ by $K(\eta(t))C(\zeta(t))$ in Lemma 1. Although Lemma 3 is not applicable due to different stability conditions in this case, an alternative parameterization result of matrix inequality can be applied and similar results can be derived [27, p. 29, Theorem 2.3.12]. However, for the general case of $D(\zeta(t)) \neq 0$, the stability conditions of the closed-loop system will contain matrix inverse terms involving $K(\eta(t))$. This is a major hurdle for extending the current results.

6. Analysis of stabilizing controller set

In this section, the stabilizing controller set is analyzed based on its connections with the standard linear quadratic regulator (LQR) problem. To see this relationship, the FTCS model is converted to the form of a JLS by representing the behaviors of two Markov processes into one, called the integrated Markov process $\phi(t)$ [20]; the solutions of the LQR problem in this JLS form are then compared with the results in Section 4.

For the basic case of an FTCS, the augmented state space of $\phi(t)$ is $S_3 = S_2 \times S_1 = \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}$, where the first element represents the FDI mode in S_2 and the second the fault mode in S_1. Let $\gamma_{(ij)(kl)}$ denote the transition rate of $\phi(t)$, which determines the transition probability of $\phi(t)$ from the augmented state (i, j) to (k, l) as shown in the following equation:

$$\phi(t) : p_{(ij)(kl)}(\triangle t) = \begin{cases} \gamma_{(ij)(kl)}\triangle t + o(\triangle t), & i \neq j,\ k \neq l; \\ 1 + \gamma_{(ij)(kl)}\triangle t + o(\triangle t), & i = j,\ k = l. \end{cases}$$

As shown in [20], $\gamma_{(ij)(kl)}$ can be derived from the transition rates of $\zeta(t)$ and $\eta(t)$:

$$\gamma_{(ij)(kl)} = \begin{cases} \alpha_{jj} + \beta^j_{ii}, & i = k,\ j = l; \\ \beta^j_{ik}, & i \neq k,\ j = l; \\ \alpha_{jl}, & i = k,\ j \neq l; \\ 0, & i \neq k,\ j \neq l. \end{cases} \tag{33}$$

For the basic case, the generator matrix F_ϕ of $\phi(t)$ is given by

$$F_\phi \triangleq [\gamma_{(ij)(kl)}]_{4\times 4} = \begin{bmatrix} (\alpha_{00} + \beta^0_{00}) & \alpha_{01} & \beta^0_{01} & 0 \\ \alpha_{10} & (\alpha_{11} + \beta^1_{00}) & 0 & \beta^1_{01} \\ \beta^0_{10} & 0 & (\alpha_{00} + \beta^0_{11}) & \alpha_{01} \\ 0 & \alpha_{10} & \beta^1_{10} & (\alpha_{11} + \beta^1_{11}) \end{bmatrix}.$$

By replacing $\zeta(t)$ and $\eta(t)$ with $\phi(t)$ in (1)–(2), the FTCS model becomes a standard JLS model:

$$\dot{x}(t) = A(\phi(t))x(t) + B(\phi(t))u(\phi(t), t), \tag{34}$$

$$y(t) = C(\phi(t))x(t) + D(\phi(t))u(\phi(t), t). \tag{35}$$

The infinite-time LQR problem of JLSs aims to find a state-feedback controller to minimize the following objective:

$$J(t_0, x(t_0), u(t)) = E\left\{\int_{t_0}^{\infty} [x^T(t)S(\phi(t))x(t) + u^T(\phi(t), t)R(\phi(t))u(\phi(t), t)]dt | x(t_0), \phi(t_0)\right\}, \tag{36}$$

where $S(\phi(t))$ and $R(\phi(t))$ denote state and control weighting matrices. For $\phi(t) = (i, j)$, denote $A_{ij} \triangleq A(\phi(t))$, $B_{ij} \triangleq B(\phi(t))$, $C_{ij} \triangleq C(\phi(t))$, $D_{ij} \triangleq D(\phi(t))$, $u_{ij}(t) \triangleq u(\phi(t), t)$, $S_{ij} \triangleq S(\phi(t))$, and $R_{ij} \triangleq R(\phi(t))$. As system matrices depend on fault mode only, $A_{ij} = A_j$, $B_{ij} = B_j$, $C_{ij} = C_j$, and $D_{ij} = D_j$.

By using state-feedback controls in a switching structure, the LQR problem was solved by Theorem 5 in [18], which stated that the optimal state-feedback controller is

$$u_{ij}(t) = -R_{ij}^{-1}B_j^T P_{ij}x(t), (i, j) \in S_3, \tag{37}$$

where $P_{ij} > 0$ satisfies the coupled algebraic Riccati equations (AREs):

$$A_j^T P_{ij} + P_{ij}A_j - P_{ij}B_jR_{ij}^{-1}B_j^T P_{ij} + \gamma_{(ij)(ij)}P_{ij} + \sum_{(k,l)\neq(i,j)} \gamma_{(ij)(kl)}P_{kl} + S_{ij} = 0, \tag{38}$$

where $(i, j), (k, l) \in S_3$.

In JLSs, the number of switching controllers is equal to that of integrated Markov states of $\phi(t)$. For this JLS model in (34)–(35) converted from an FTCS model, there are four controllers designed corresponding to four states of $\phi(t)$ as given in (37). When $\phi(t) = (i, j)$, the following state-feedback gain is in use:

$$K_{ij} = -R_{ij}^{-1}B_j^T P_{ij}, \qquad (i, j) \in S_3. \tag{39}$$

In contrast, in FTCSs, the number of switching controllers is equal to that of fault modes, so only two controllers exist for the basic case in (34)–(35). Therefore, JLS's have more design freedom whereas FTCSs are more restrictive, and the design methods of JLSs are not applicable to FTCSs. But the controller designed in FTCSs can be analyzed by the methods in JLSs considering that two controllers can be deemed a special case of two pairs of identical controllers. For example, K_0 and K_1 in FTCSs are deemed K_0, K_0, K_1, and K_1 in JLSs.

Proposition 1. *(20)–(23) in Theorem 1 are equivalent to AREs (38) of the LQR problem in JLSs. In other words,* $\mathbf{P} = \{P_{ij},\ i, j \in \{0, 1\}\}$ *satisfies Theorem 1 if* *and only if it is a feasible solution of AREs (38) corresponding to the following LQR weighting matrices:*

$$S_{ij} = \mu_{ij} P_{ij} B_j B_j^T P_{ij} - (\bar{A}_{ij}^T P_{ij} + P_{ij} \bar{A}_{ij} + \beta_{i(1-i)}^j P_{(1-i)j} + \alpha_{j(1-j)} P_{i(1-j)}), \tag{40}$$

$$R_{ij} = 1/\mu_{ij}, \qquad (i, j) \in S_3. \tag{41}$$

Proposition 2. *The parameterization set $\mathcal{W}_{ij}$ in (27) contains an LQR controller of JLSs given in (37) corresponding to the weighting matrices in (40)–(41).*

These two propositions are derived from Theorem 1 and Lemma 3, and the proofs are given in the Appendix. Proposition 2 shows that $\mathcal{W}_{00}$ and $\mathcal{W}_{01}$ contain an LQR controller of JLSs, and these sets are around an LQR controller; so the parameterization set $\mathcal{K}_0^{\mathbf{P}}$ is also around an LQR controller. This connection provides a meaningful interpretation of the stabilizing controller set found in Section 5.

7. Synthesis of generator matrices

Clearly, the generator matrices of $\zeta(t)$ and $\eta(t)$ are crucial parameters in the model of FTCSs. In this section, synthesis methods are presented based on two structures of Markov processes and the knowledge of failure rates and FDI history data.

Let $Y(t)$ denote a homogeneous continuous-time Markov process in a finite state space S_Y. Let $T_0, T_1, T_2, \cdots$ denote transition times and $Y_0, Y_1, Y_2, \cdots$ the successive states visited by $Y(t)$. If $Y_n = i$, $[T_n, T_{n+1})$ is called the sojourn interval, and $T_{n+1} - T_n$ the sojourn time at state i, $i \in S_Y$, $n \in \mathbb{N}$. Markov process theory states that $\{Y_n, n \in \mathbb{N}\}$ forms a Markov chain, and $T_{n+1} - T_n$ follows an exponential distribution with a parameter depending on Y_n only [5, Chap. 8]. This is the first structure of a Markov process.

Let Q_Y denote the generator matrix of $Y(t)$ and $Q_Y(i, j)$ its transition rate. The transition probabilities of Y_n are

$$\Pr\{Y_{n+1} = j | Y_n = i\} = \frac{Q_Y(i, j)}{-Q_Y(i, i)}, \qquad i \neq j, \tag{42}$$

and $\Pr\{Y_{n+1} = i | Y_n = i\} = 0$, $i, j \in S_Y$. If $Q_Y(i, i) = 0$, state i is absorbing, and $\Pr\{Y_{n+1} = j | Y_n = i\} = 0$ for all $j \in S_Y$. The sojourn time distribution at state i is

$$\Pr\{T_{n+1} - T_n > t | Y_n = i\} = e^{Q_Y(i,i)t}. \tag{43}$$

Note that $Q_Y(i, j) \geq 0$, and $Q_Y(i, i) = -\sum_{j \in S_Y} Q_Y(i, j)$, $i \neq j$.

The second structure uses competitions among independent exponential random variables to determine sojourn times and successive transition states. When $Y(t) = i$, an exponentially distributed random variable τ_{ij} with rate $Q_Y(i, j)$

is associated with the transition to j in S_Y. The transition can be viewed as a competition process among τ_{ij}, $j \in S_Y$: the state associated with the minimum of τ_{ij} is the successive state visited by $Y(t)$, and this minimum value gives the sojourn time at i. Based on the property of independent exponential random variables [24, p. 243], (42) and (43) can be derived under this structure.

Using the Markov process $\zeta(t)$ to describe fault occurrences requires the assumption of constant failure rates, or equivalently, exponential distribution of lifetimes, which is generally valid for the majority of component lifetimes [19]. The generator matrix of $\zeta(t)$ can be synthesized based on the second structure and failure rates. In the state space S_1 of $\zeta(t)$, 0 usually represents fault-free mode, and other states describe specific faults and may also describe their combinations. The transitions of $\zeta(t)$ may represent fault occurrences, repairs, or recoveries from intermediate faults depending on transition modes and directions.

For example, for a system with two types of faults, S_1 can be defined as $\{0, 1, 2, 3\}$, where each mode represents respectively fault-free mode, fault type I, fault type II, and their simultaneous occurrences. The transitions from mode 0 to 1 or 2 represent the occurrences of fault type I or II respectively, whereas the transitions of opposite directions represent repairs or recoveries from these faults. In cases of multiple faults that may occur at a particular mode, there exist competitions among exponential lifetime random variables: the fault occurring first with minimum lifetime makes $\zeta(t)$ jump to the corresponding mode in S_1, and the minimum lifetime gives its sojourn time. So, the transition rates in the upper triangular part of F correspond to failure rates; and those in the lower triangular part represent the rates of repairs or recoveries. Let the failure rates of two faults be denoted by λ_1 and λ_2 respectively, and the generator matrix of $\zeta(t)$ is

$$F = \begin{bmatrix} -(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\ 0 & -\lambda_2 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the transition rates in the lower triangular part are all zeros as no repair or intermediate fault is assumed.

$\eta(t)$ models FDI results, and its state space S_2 is usually identical to S_1. Its generator matrix can be estimated using FDI history data based on the first structure of Markov processes. This history data should record the transition states and sojourn times of FDI under known fault modes, which can be obtained by experimental testing of FDI schemes. From (42)–(43), it suffices to estimate the transition probabilities and the means of sojourn time distributions in order to determine the generator matrix of $\eta(t)$.

When $\zeta(t) = k$ and $\eta(t) = i$, the sample sojourn time of $\eta(t)$ at i is recorded as $\tau_i^{(l)}, l = 1, 2, \ldots, N$. The sample average

$$\bar{\tau}_i = -\sum_{l=1}^{N} \tau_i^{(l)}/N$$

converges to $1/\beta_{ii}^k$ in probability 1 as $N \to \infty$ based on the law of large numbers and (43). Let $\hat{\beta}_{ii}^k = 1/\bar{\tau}_i$ denote the estimate of β_{ii}^k. If there is no transition from state i for $\eta(t)$, this state is deemed to be absorbing, and $\hat{\beta}_{ii}^k = 0$ in this case.

The transition probability can be estimated by transition frequencies. If there are M transitions of $\eta(t)$ to mode j within N transitions leaving i in FDI history data, the transition frequency M/N converges to transition probability with probability 1 as $N \to \infty$. Using (42), the transition rate from i to j is estimated as

$$\hat{\beta}_{ij}^k = -\hat{\beta}_{ii}^k M/N.$$

Using this method, all elements in the generator matrix of $\eta(t)$ can be estimated. Moreover, as in (31), to ensure specific estimate precisions, the lower bound of sample quantity N can be determined using Chernoff's bound.

Remark 6. Fault effects on system dynamics are described by different system matrices in the dynamic model (1)–(2), $A(\zeta(t))$, $B(\zeta(t))$, $C(\zeta(t))$, and $D(\zeta(t))$ depending on $\zeta(t)$. The FDI scheme can be designed by standard model-based methods using these dynamic models [15]. Although some iterative algorithms exist to obtain a sequence estimate of Markov states based on the probabilistic description of system modes, the computational cost is not suitable for online implementation and controller reconfiguration, and the algorithms are designed for a discrete-time Markov chain only [6]. The transition characteristics of the FDI mode can be described by a Markov process from the perspective of closed-loop stability of the reconfigured system [23]. But it is necessary to have FDI history data available for estimating Markov transition rates.

8. An illustrative example

Consider a longitudinal vertical takeoff and landing aircraft model in the form of (1)–(2) with the following system matrices [38]. The subscript 0 represents the fault-free mode and 1 the faulty mode. In the faulty mode, the actuator failure is considered, and the effectiveness of the first actuator is reduced by half as reflected in B_1.

$$A_0 = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.420 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}, \qquad A_1 = A_0,$$

$$B_0 = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.2211 & 0.1761 \\ 1.7723 & -7.5922 \\ -2.76 & 4.49 \\ 0 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \qquad C_1 = C_0.$$

The generator matrices of $\zeta(t)$ and $\eta(t)$ are

$$F = \begin{bmatrix} -0.0017 & 0.0017 \\ 0 & 0 \end{bmatrix}, \qquad H^0 = \begin{bmatrix} -0.0204 & 0.0204 \\ 3.9039 & -3.9039 \end{bmatrix},$$

$$H^1 = \begin{bmatrix} -2.9925 & 2.9925 \\ 0.0515 & -0.0515 \end{bmatrix}.$$

According to F, the mean lifetime before fault occurrence is $1/0.0017 = 588.24$ seconds, and the fault mode is absorbing as shown in the second zero row of F, i.e., there is no repair or recovery from the intermediate fault. For FDI, according to the first row of H^0, when the system is in the fault-free mode, the mean time of a false alarm is $1/0.0204 = 49.02$ seconds; and according to its second row, the mean time of returning to correct detection after a false alarm is $1/3.9039 = 0.2562$ of a second. H^1 can be interpreted similarly: the mean time of a missing detection is $1/0.0515 = 19.42$ seconds, and the mean time of returning to correct detection after a missing detection is $1/2.9925 = 0.3342$ of a second.

Solve the conditions in Theorem 1 for P_{ij} as follows:

$$P_{00} = \begin{bmatrix} 0.0114 & 0.0009 & -0.0028 & -0.0065 \\ 0.0009 & 0.0043 & -0.0011 & 0.0004 \\ -0.0028 & -0.0011 & 0.0099 & 0.0079 \\ -0.0065 & 0.0004 & 0.0079 & 0.0208 \end{bmatrix},$$

$$P_{01} = \begin{bmatrix} 3.5840 & 0.1916 & -0.5806 & -1.1955 \\ 0.1916 & 3.2196 & -0.2804 & -0.0447 \\ -0.5806 & -0.2804 & 3.4266 & 1.1760 \\ -1.1955 & -0.0447 & 1.1760 & 4.9369 \end{bmatrix},$$

$$P_{10} = \begin{bmatrix} 0.0484 & 0.0050 & -0.0073 & -0.0084 \\ 0.0050 & 0.0619 & -0.0147 & 0.0052 \\ -0.0073 & -0.0147 & 0.0703 & 0.0102 \\ -0.0084 & 0.0052 & 0.0102 & 0.0669 \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} 2.7515 & -0.1065 & -1.0816 & -1.2975 \\ -0.1065 & 3.5939 & -0.1549 & -0.0833 \\ -1.0816 & -0.1549 & 3.4801 & 1.6071 \\ -1.2975 & -0.0833 & 1.6071 & 3.8389 \end{bmatrix}.$$

Based on Proposition 1, these P_{ij} correspond to the following LQR weighting matrices S_{ij} and R_{ij}:

$$S_{00} = \begin{bmatrix} 0.0120 & 0.0184 & -0.0303 & -0.0113 \\ 0.0184 & 0.0351 & -0.0533 & -0.0144 \\ -0.0303 & -0.0533 & 0.0884 & 0.0279 \\ -0.0113 & -0.0144 & 0.0279 & 0.0165 \end{bmatrix},$$

$$S_{01} = \begin{bmatrix} 213.7 & 1161 & -954.4 & -307.4 \\ 1161.0 & 7689.3 & -5618.3 & -1764.3 \\ -954.4 & -5618.3 & 4436.9 & 1416.5 \\ -307.4 & -1764.3 & 1416.5 & 460.1 \end{bmatrix},$$

$$S_{10} = \begin{bmatrix} 0.2133 & 0.3874 & -0.4153 & 0.0212 \\ 0.3874 & 2.7692 & -2.4173 & 0.1968 \\ -0.4153 & -2.4173 & 2.7225 & -0.0830 \\ 0.0212 & 0.1968 & -0.0830 & 0.1956 \end{bmatrix},$$

$$S_{11} = \begin{bmatrix} 267.1 & 1348.6 & -1027.9 & -478 \\ 1348.6 & 9117.3 & -5859.5 & -2690.3 \\ -1027.9 & -5859.5 & 4154.7 & 1921.8 \\ -478 & -2690.3 & 1921.8 & 891.3 \end{bmatrix},$$

$$R_{00} = 0.0676, \qquad R_{01} = 0.0913, \qquad R_{10} = 0.1570, \qquad R_{11} = 0.0911.$$

Following the design procedure with $\epsilon = 0.02$ and $\delta = 0.02$, 194 sample controllers are generated and evaluated with respect to MTTF. It is found that the following approximately optimal controller $\hat{\mathbf{K}}^* = \{\hat{K}_0^*,\ \hat{K}_1^*\}$ achieves MTTF = 197.3208 seconds with $\Pr\{\psi(\mathbf{K}^*) - \psi(\hat{\mathbf{K}}^*) \geq 0.02\} \leq 0.02$, where $\psi(\mathbf{K}^*)$ denotes the optimal MTTF with the optimal controller $\mathbf{K}^*$:

$$\hat{K}_0^* = \begin{bmatrix} -0.6566 & -0.7359 & 2.0731 & 1.1449 \\ 0.4176 & 1.3777 & -1.1316 & -1.0322 \end{bmatrix},$$

$$\hat{K}_1^* = \begin{bmatrix} -0.1117 & 0.2114 & 0.1399 & 0.4621 \\ 0.0621 & 0.5747 & -0.1667 & -0.3248 \end{bmatrix}.$$

For comparison, arbitrarily select another stabilizing controller $\mathbf{K} = \{K_0,\ K_1\}$ with MTTF = 55.8319 seconds:

$$K_0 = \begin{bmatrix} -3.2572 & -1.8991 & 8.0921 & 6.7639 \\ -0.8941 & 0.8646 & 1.0997 & 1.1335 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.0272 & 0.2312 & 0.0945 & 0.0146 \\ 0.0427 & -0.0603 & -0.0534 & -0.5202 \end{bmatrix}.$$

To compare the time domain performance of these two controllers, a white noise disturbance is applied to the system. With initial state $x(0) = [2\ \ -2\ \ 2\ \ -2]^T$, output trajectories are shown in Figures 4 and 5, where $\zeta(t)$ remains at fault-free mode 0, and $\eta(t)$ is manually set to 1 such that FDI gives false alarms when

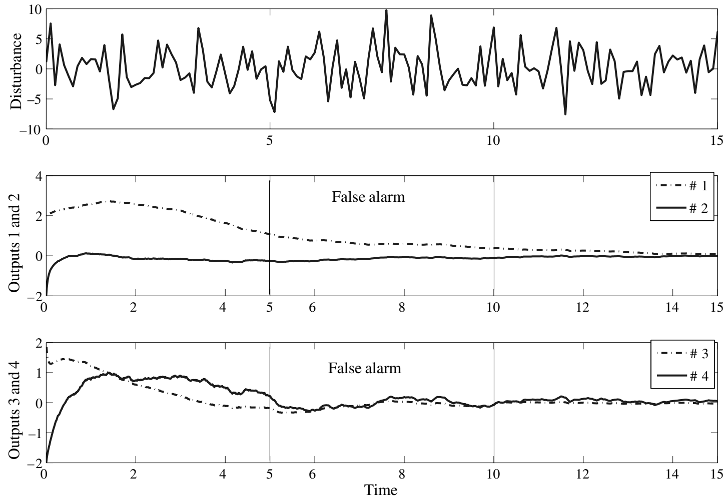

Figure 4. Output trajectories when using $\hat{\mathbf{K}}^{*}$.

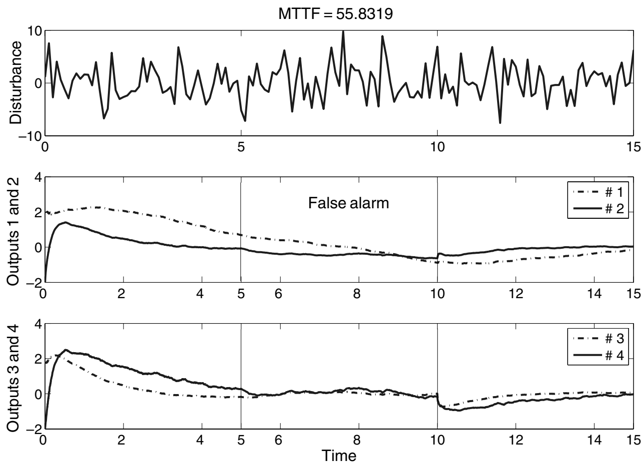

Figure 5. Output trajectories when using **K**.

$5 \leq t < 10$. In fact, the sample paths of $\zeta(t)$ and $\eta(t)$ can be generated based on their generator matrices. But, the possibility of observing fault occurrences or false alarms in a short time is very small. In order to study system responses under false alarms, we manually set the transitions of $\eta(t)$. Moreover, to examine the

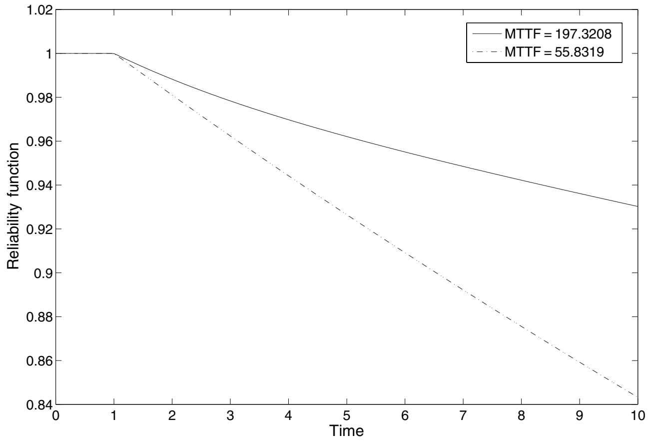

Figure 6. Compare reliability functions when using $\hat{\mathbf{K}}^*$ and $\mathbf{K}$.

robust performance of controllers, system matrices are perturbed probabilistically around their nominal values during the simulation.

As shown in Figures 4 and 5, output trajectories are converging and disturbances attenuated by both controllers; overall, $\hat{\mathbf{K}}^*$ seems to have better disturbance attenuation effects. This can be further validated by comparing the closed-loop H_∞ norms. For $\hat{\mathbf{K}}^*$, the nominal closed-loop H_∞ norm is 0.1294 when $\eta(t) = 0$ and 0.1565 when $\eta(t) = 1$; for $\mathbf{K}$, it is 0.1088 when $\eta(t) = 0$ and 0.2178 when $\eta(t) = 1$. If probabilistic modeling uncertainties are considered, for $\hat{\mathbf{K}}^*$, the probabilities that the H_∞ norm is no greater than 1 are 0.6467 and 0.7600 when $\eta(t) = 0$ and 1 respectively [31]; for $\mathbf{K}$, the probabilities are 0.6328 and 0.1043 respectively, much worse than $\hat{\mathbf{K}}^*$, especially under false alarms. This finding is not surprising because in this example the H_∞ norm under probabilistic uncertainties has been used as a control objective in the definition of a reliability function. The case for missing detection of FDI under fault occurrence can be studied in a similar way, which is not included for brevity.

The reliability functions of FTCSs for these two controllers are shown in Figure 6, and the reliability shows great improvement by using $\hat{\mathbf{K}}^*$. To test the statement $\Pr\{\psi(\mathbf{K}^*) - \psi(\hat{\mathbf{K}}) > 0.02\} \leq 0.02$, 1000 stabilizing controller samples are generated, and the MTTF of the FTCS for each controller is calculated, as shown in Figure 7. From this figure, it is found that only one controller has better MTTF than $\hat{\mathbf{K}}^*$. Therefore, the randomized algorithm gives a valid estimate of the optimum with the specified precision.

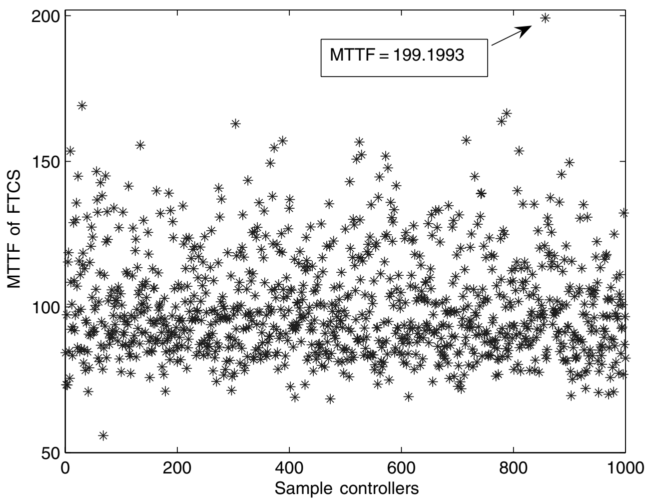

Figure 7. MTTF of FTCS for 1000 generated stabilizing controllers.

9. Conclusion

This paper presents a probabilistic design method of FTCSs based on the stability and reliability criteria. The basic idea is to develop a stabilizing controller parameterization set and to apply the randomized algorithms to find the statistically optimal controller in terms of system reliability. The stabilization conditions are given in the form of LMIs, and the free parameters in the controller parameterization set are real matrices and scalars, which is convenient for numerical implementation. An example is presented, and the results show that a statistically optimal controller with highest reliability can be obtained by this method.

Appendix

Proof of Lemma 3. The equivalence between statements 1 and 2 is a special form of the well-known Projection Lemma [2]. Here, we prove (7) only. When statements 1 and 2 hold, it is equivalent to

$$MX + (MX)^T + \rho X^T X < -Q, \tag{44}$$

for some scalar $\rho > 0$. Add $\rho^{-1}MM^T$ to both sides, complete the square on the left-hand side of (44), and we have

$$(\rho^{-1}M + X^T)\rho(\rho^{-1}M^T + X) < \rho^{-1}MM^T - Q. \tag{45}$$

Obviously, (45) holds if and only if $\rho^{-1}MM^T - Q > 0$ as the left-hand side of (45) is positive semidefinite. By taking the matrix square root, (45) is equivalent to

$$(\rho^{-1}MM^T - Q)^{-1/2}(M\rho^{-1} + X^T)\rho(\rho^{-1}M^T + X)(\rho^{-1}MM^T - Q)^{-1/2} < I. \tag{46}$$

Define $L \triangleq \rho^{1/2}(\rho^{-1}M^T + X)(\rho^{-1}MM^T - Q)^{-1/2}$. Then $\|L\| < 1$ and

$$X = -\rho^{-1}M^T + \rho^{-1/2}L(\rho^{-1}MM^T - Q)^{1/2}.$$

To determine the upper bound of ρ, convert (44) to the following matrix inequality by the Schur's complement lemma [2]:

$$\begin{bmatrix} -\rho^{-1}I & X \\ X^T & MX + (MX)^T + Q \end{bmatrix} < 0.$$

Define a new decision variable $a \triangleq \rho^{-1} > 0$, and the minimum value of a gives the upper bound $\rho_{\max}$. Moreover, $\rho \in (0,\ \rho_{\max})$ ensures $\rho^{-1}MM^T - Q > 0$ from (45). □

Proof of Lemma 4. Based on Lemma 3, each inequality in (11)–(14) has feasible solution K_0 or K_1 if and only if the corresponding condition in (16)–(19) holds. Considering that (11)–(14) must hold simultaneously for system stability, (16)–(19) are only necessary conditions. If B_0 has full row rank, (11) and (12) always have feasible solutions for any P_{00}, P_{10}, Q_{00}, and Q_{10}, so (16) and (17) are removed; similarly, if B_1 has full row rank, (18) and (19) are removed. □

Proof of Proposition 1. Substitute the system parameters into (38); we obtain four coupled AREs. Note that the system matrices depend on the fault mode only, the second element of $\phi(t)$. For example, $A_{ij} = A_j$. Let us consider the following ARE for $\phi(t) = (0, 0)$:

$$A_0^T P_{00} + P_{00}A_0 - P_{00}B_0R_{00}^{-1}B_0^T P_{00} - (\alpha_{01} + \beta_{01}^0)P_{00} + \beta_{01}^0 P_{01} + \alpha_{01}P_{10} + S_{00} = 0.$$

Use $\bar{A}_{00} = A_0 - 0.5\beta_{01}^0 - 0.5\alpha_{01}$ defined in Theorem 1 to simplify this equation, and we have

$$\bar{A}_{00}^T P_{00} + P_{00}\bar{A}_{00} + \beta_{01}^0 P_{01} + \alpha_{01}P_{10} + S_{00} = P_{00}B_0R_{00}^{-1}B_0^T P_{00}. \tag{47}$$

Let $R_{00} = 1/\mu_{00}$ and compare (47) with (25). If (47) holds, (25) obviously holds considering $S_{00} > 0$; if (25) holds, (47) also holds with

$$S_{00} = \mu_{00}P_{00}B_0B_0^T P_{00} - (\bar{A}_{00}^T P_{00} + P_{00}\bar{A}_{00} + \beta_{01}^0 P_{10} + \alpha_{01}P_{01})$$

$$= P_{00}B_0R_{00}^{-1}B_0^T P_{00} - (\bar{A}_{00}^T P_{00} + P_{00}\bar{A}_{00} + \beta_{01}^0 P_{10} + \alpha_{01}P_{01}) > 0.$$

So, (47) and (25) are equivalent. It immediately follows that (47) and (11) are equivalent. Similarly, we can establish the equivalence between (12)–(14) and the other three AREs of (38) corresponding to $\phi(t) = (0, 1), (1, 0), (1, 1)$. □

Proof of Proposition 2. Recall (27) and Lemma 3; if Theorem 1 holds, the feasible solutions for each inequality in (11)–(14) are parameterized by

$$\mathcal{W}_{ij} = \{K'_{ij} | K'_{ij} = -\rho_{ij}^{-1} B_j^T P_{ij} + \rho_{ij}^{-1/2} L_{ij} (\rho_{ij}^{-1} P_{ij} B_j B_j^T P_{ij} - Q_{ij})^{1/2},$$

$$\|L_{ij}\| < 1, \qquad \rho_{ij} \in (0,\ \rho_{ij\max})\}, \qquad i \in S_2,\ j \in S_1, \tag{48}$$

where L_{ij} and ρ_{ij} are free parameters and $\rho_{ij\max}$ is calculated by (8) in Lemma 3. Furthermore, by Lemma 3, $\rho_{ij} \in (0,\ \rho_{ij\max})$ if and only if it satisfies (9): $\rho_{ij}^{-1} P_{ij} B_j B_j^T P_{ij} - Q_{ij} > 0$.

ρ_{ij} may take the value of μ_{ij}^{-1} because

$$\mu_{ij} P_{ij} B_j B_j^T P_{ij} > Q_{ij}, \qquad i \in S_2,\ j \in S_1. \tag{49}$$

To see this inequality (49), take $i = 0$ and $j = 0$ as an example, and substitute the definition of Q_{00} in (15). (49) then becomes (25), which has been proved in Theorem 1.

Let the free parameter $L_{ij} = 0$ and the corresponding element in $\mathcal{W}_{ij}$ is

$$K'_{ij} = -\mu_{ij} B_j^T P_{ij}, \qquad i \in S_2,\ j \in S_1. \tag{50}$$

Considering (41) in Proposition 1,

$$K'_{ij} = -R_{ij}^{-1} B_j^T P_{ij}, \qquad i \in S_2,\ j \in S_1, \tag{51}$$

which is obviously the LQR controller in (39). □

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